Integration Using Trigonometric Identities

When the integrand consists of trigonometric functions, we use known identities to convert it into a form which can easily be integrated. Some of the identities useful for this purpose are given below:

(i) $2\sin^2\left(\frac{x}{2}\right) = (1 - \cos x)$ (ii) $2\cos^2\left(\frac{x}{2}\right) = (1 + \cos x)$

(iii) $2\sin a \cos b = \sin (a + b) + \sin (a - b)$

(iv) $2\cos a \sin b = \sin (a + b) - \sin (a - b)$

(v) $2\cos a\cos b = \cos(a+b) + \cos(a-b)$

(vi) $2\sin a\sin b = \cos(a-b) - \cos(a+b)$

SOLVED EXAMPLES

EXAMPLE 1 *Evaluate:* *(i)* $\int \sin^2\frac{x}{2}dx$ *(ii)* $\int \tan^2\frac{x}{2}dx$

(iii) $\int \cos^2 nx\,dx$ *(iv)* $\int \cos^5 x\,dx$

(v) $\int \sin^7 x\,dx$ *(vi)* $\int \sin^3(2x+1)\,dx$

SOLUTION (i) $\int \sin^2\frac{x}{2}dx = \frac{1}{2}\int 2\sin^2\frac{x}{2}dx$

$$= \frac{1}{2}\int(1-\cos x)\,dx = \frac{1}{2}\int dx - \frac{1}{2}\int \cos x\,dx$$

$$= \frac{1}{2}x - \frac{1}{2}\sin x + C.$$

(ii) $\int \tan^2\frac{x}{2}dx = \int\left(\sec^2\frac{x}{2} - 1\right)dx = \int \sec^2\frac{x}{2}dx - \int dx$

$$= 2\int \sec^2 t\,dt - \int dx, \text{ where } \frac{x}{2} = t$$

$$= 2\tan t - x + C = 2\tan\frac{x}{2} - x + C.$$

(iii) $\int \cos^2 nx\,dx = \frac{1}{2}\int 2\cos^2 nx\,dx$

$$= \frac{1}{2}\int(1+\cos 2nx)\,dx = \frac{1}{2}\int dx + \frac{1}{2}\int \cos 2nx\,dx$$

$$= \frac{x}{2} + \frac{1}{4n}\sin 2nx + C.$$

(iv) $\int \cos^5 x\,dx = \int \cos^4 x\cdot\cos x\,dx$

$$= \int(1-\sin^2 x)^2\cdot\cos x\,dx = \int(1-t^2)^2\,dt, \text{ where } \sin x = t$$

$$= \int(1+t^4-2t^2)\,dt = \int dt + \int t^4 dt - 2\int t^2 dt$$

$$= t + \frac{t^5}{5} - \frac{2t^3}{3} + C = \sin x + \frac{1}{5}\sin^5 x - \frac{2}{3}\sin^3 x + C.$$

(v) $\int \sin^7 x\,dx = \int \sin^6 x\cdot\sin x\,dx$

$$= \int(1-\cos^2 x)^3\sin x\,dx$$

$$= -\int(1-t^2)^3\,dt, \text{ where } \cos x = t$$

$$= \int(t^6 - 3t^4 + 3t^2 - 1)\,dt = \frac{t^7}{7} - \frac{3t^5}{5} + t^3 - t + C$$

$$= \frac{1}{7}\cos^7 x - \frac{3}{5}\cos^5 x + \cos^3 x - \cos x + C.$$

(vi) $\int \sin^3(2x+1)\,dx = \int \{1-\cos^2(2x+1)\}\cdot \sin(2x+1)\,dx$

$$= -\frac{1}{2}\int (1-t^2)\,dt, \quad \text{where } \cos(2x+1) = t$$

$$= -\frac{1}{2}\int dt + \frac{1}{2}\int t^2 dt = -\frac{1}{2}t + \frac{1}{6}t^3 + C$$

$$= -\frac{1}{2}\cos(2x+1) + \frac{1}{6}\cos^3(2x+1) + C.$$

EXAMPLE 2 *Evaluate* $\int \cos mx \cos nx\,dx$, *when (i)* $m \neq n$ *(ii)* $m = n$.

SOLUTION (i) When $m \neq n$, we have

$$\int \cos mx \cos nx\,dx = \frac{1}{2}\int [\cos(m+n)x + \cos(m-n)x]\,dx$$

$$= \frac{1}{2}\int \cos(m+n)x\,dx + \frac{1}{2}\int \cos(m-n)x\,dx$$

$$= \frac{\sin(m+n)x}{2(m+n)} + \frac{\sin(m-n)x}{2(m-n)} + C.$$

(ii) When $m = n$, we have

$$\int \cos mx \cos nx\,dx = \int \cos^2 nx\,dx$$

$$= \frac{1}{2}\int 2\cos^2 nx\,dx = \frac{1}{2}\int (1+\cos 2nx)\,dx$$

$$= \frac{1}{2}\int dx + \frac{1}{2}\int \cos 2nx\,dx = \frac{x}{2} + \frac{\sin 2nx}{4n} + C.$$

EXAMPLE 3 *Evaluate:*

(i) $\int \sin 3x \sin 2x\,dx$ *(ii)* $\int \cos 3x \sin 2x\,dx$

(iii) $\int \cos 4x \cos x\,dx$ **[CBSE 2001]** *(iv)* $\int \sin^3 x \cos^3 x\,dx$

SOLUTION (i) Using $2\sin a \sin b = \cos(a-b) - \cos(a+b)$, we have

$$\int \sin 3x \sin 2x\,dx = \frac{1}{2}\int 2\sin 3x \sin 2x\,dx$$

$$= \frac{1}{2}\int (\cos x - \cos 5x)\,dx$$

$$= \frac{1}{2}\int \cos x\,dx - \frac{1}{2}\int \cos 5x\,dx$$

$$= \frac{1}{2}\sin x - \frac{\sin 5x}{10} + C.$$

(ii) Using $2\cos a \sin b = \sin(a+b) - \sin(a-b)$, we get

$$\int \cos 3x \sin 2x\,dx = \frac{1}{2}\int 2\cos 3x \sin 2x\,dx$$

$$= \frac{1}{2}\int (\sin 5x - \sin x)\,dx$$

$$= \frac{1}{2}\int \sin 5x\, dx - \frac{1}{2}\int \sin x\, dx$$

$$= \frac{-\cos 5x}{10} + \frac{\cos x}{2} + C.$$

(iii) Using $2\cos a \cos b = \cos(a+b) + \cos(a-b)$, we get

$$\int \cos 4x \cos x\, dx = \frac{1}{2}\int 2\cos 4x \cos x\, dx$$

$$= \frac{1}{2}\int (\cos 5x + \cos 3x)\, dx$$

$$= \frac{1}{2}\int \cos 5x\, dx + \frac{1}{2}\int \cos 3x\, dx$$

$$= \frac{\sin 5x}{10} + \frac{\sin 3x}{6} + C.$$

(iv) $$\int \sin^3 x \cos^3 x\, dx = \int \sin^3 x \cos^2 x \cos x\, dx$$

$$= \int \sin^3 x(1 - \sin^2 x)\cos x\, dx$$

$$= \int t^3(1 - t^2)\, dt, \quad \text{where } \sin x = t$$

$$= \int t^3 dt - \int t^5 dt = \frac{t^4}{4} - \frac{t^6}{6} + C$$

$$= \frac{1}{4}\sin^4 x - \frac{1}{6}\sin^6 x + C.$$

EXAMPLE 4 *Evaluate* $\int \cos x \cos 2x \cos 3x\, dx.$

SOLUTION $\int \cos x \cos 2x \cos 3x\, dx$

$$= \frac{1}{2}\int (2\cos x \cos 2x)\cos 3x\, dx$$

$$= \frac{1}{2}\int (\cos 3x + \cos x)\cos 3x\, dx = \frac{1}{2}\int (\cos^2 3x + \cos x \cos 3x)\, dx$$

$$= \frac{1}{4}\int (2\cos^2 3x)\, dx + \frac{1}{4}\int (2\cos x \cos 3x)\, dx$$

$$= \frac{1}{4}\int (1 + \cos 6x)\, dx + \frac{1}{4}\int (\cos 4x + \cos 2x)\, dx$$

$$= \frac{1}{4}\int dx + \frac{1}{4}\int \cos 6x\, dx + \frac{1}{4}\int \cos 4x\, dx + \frac{1}{4}\int \cos 2x\, dx$$

$$= \frac{1}{4}x + \frac{1}{4}\cdot\frac{\sin 6x}{6} + \frac{1}{4}\cdot\frac{\sin 4x}{4} + \frac{1}{4}\cdot\frac{\sin 2x}{2} + C$$

$$= \frac{x}{4} + \frac{\sin 6x}{24} + \frac{\sin 4x}{16} + \frac{\sin 2x}{8} + C.$$

EXAMPLE 5 *Evaluate* $\int \sec^4 x \tan x\, dx.$

SOLUTION
$$\int \sec^4 x \tan x\, dx = \int \sec^2 x \cdot \sec^2 x \tan x\, dx$$
$$= \int (1 + \tan^2 x)\sec^2 x \tan x\, dx$$
$$= \int (1 + t^2) t\, dt, \text{ where } \tan x = t$$
$$= \int t\, dt + \int t^3 dt = \frac{t^2}{2} + \frac{t^4}{4} + C$$
$$= \frac{1}{2}\tan^2 x + \frac{1}{4}\tan^4 x + C.$$

EXAMPLE 6 *Evaluate* $\int \sin^4 x\, dx.$ **[CBSE 2004]**

SOLUTION
$$\int \sin^4 x\, dx = \frac{1}{4}\int (2\sin^2 x)^2 dx$$
$$= \frac{1}{4}\int (1 - \cos 2x)^2 dx = \frac{1}{4}\int (1 + \cos^2 2x - 2\cos 2x)\, dx$$
$$= \frac{1}{8}\int (2 + 2\cos^2 2x - 4\cos 2x)\, dx$$
$$= \frac{1}{8}\int [2 + (1 + \cos 4x) - 4\cos 2x]\, dx$$
$$= \frac{3}{8}\int dx + \frac{1}{8}\int \cos 4x\, dx - \frac{1}{2}\int \cos 2x\, dx$$
$$= \frac{3}{8}x + \frac{\sin 4x}{32} - \frac{\sin 2x}{4} + C.$$

EXAMPLE 7 *Evaluate* $\int \frac{\sin x}{\sin (x - \alpha)} dx.$ **[CBSE 2003C, '04]**

SOLUTION Put $(x - \alpha) = t$ so that $x = (t + \alpha)$ and $dx = dt.$

$$\therefore \quad \int \frac{\sin x}{\sin (x - \alpha)} dx = \int \frac{\sin (t + \alpha)}{\sin t} dt$$
$$= \int \frac{\sin t \cos \alpha + \cos t \sin \alpha}{\sin t} dt$$
$$= \cos \alpha \cdot \int dt + \sin \alpha \cdot \int \cot t\, dt$$
$$= \cos \alpha \cdot t + \sin \alpha \cdot \log |\sin t| + C$$
$$= (\cos \alpha)(x - \alpha) + \sin \alpha \cdot \log |\sin (x - \alpha)| + C.$$

EXAMPLE 8 *Evaluate: (i)* $\int \frac{\sin 4x}{\cos 2x} dx$ *(ii)* $\int \frac{\sin 4x}{\sin x} dx$

SOLUTION (i) $\int \frac{\sin 4x}{\cos 2x} dx = \int \frac{2\sin 2x \cos 2x}{\cos 2x} dx$
$$= 2\int \sin 2x\, dx = -\cos 2x + C.$$

(ii) $\int \frac{\sin 4x}{\sin x}\,dx = \int \frac{2\sin 2x\cos 2x}{\sin x}\,dx$

$= \int \frac{4\sin x\cos x\cos 2x}{\sin x}\,dx = 2\int 2\cos x\cos 2x\,dx$

$= 2\int (\cos 3x + \cos x)\,dx = 2\int \cos 3x\,dx + 2\int \cos x\,dx$

$= \frac{2\sin 3x}{3} + 2\sin x + C.$

EXAMPLE 9 *Evaluate* $\int \sqrt{1+\sin x}\,dx.$

SOLUTION $\int \sqrt{1+\sin x}\,dx = \int \sqrt{\sin^2\frac{x}{2} + \cos^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}}\,dx$

$= \int \left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)dx = \int \sin\frac{x}{2}\,dx + \int \cos\frac{x}{2}\,dx$

$= -2\cos\frac{x}{2} + 2\sin\frac{x}{2} + C.$

EXAMPLE 10 *Evaluate* $\int \frac{\sin^2 x}{(1+\cos x)^2}\,dx.$

SOLUTION $\int \frac{\sin^2 x}{(1+\cos x)^2}\,dx = \int \left(\frac{\sin x}{1+\cos x}\right)^2 dx = \int \left[\frac{2\sin(x/2)\cos(x/2)}{2\cos^2(x/2)}\right]^2 dx$

$= \int \tan^2(x/2)\,dx = \int \left(\sec^2\frac{x}{2} - 1\right)dx$

$= \int \sec^2(x/2)\,dx - \int dx = 2\tan(x/2) - x + C.$

EXAMPLE 11 *Evaluate:* *(i)* $\int \sin x\sqrt{1-\cos 2x}\,dx$ *(ii)* $\int \frac{\cos 2x}{\sqrt{1+\cos 4x}}\,dx$

SOLUTION (i) $\int \sin x\sqrt{1-\cos 2x}\,dx$

$= \int \sin x\cdot\sqrt{2\sin^2 x}\,dx = \sqrt{2}\int \sin^2 x\,dx = \frac{1}{\sqrt{2}}\int 2\sin^2 x\,dx$

$= \frac{1}{\sqrt{2}}\int (1-\cos 2x)\,dx = \frac{1}{\sqrt{2}}\int dx - \frac{1}{\sqrt{2}}\int \cos 2x\,dx$

$= \frac{1}{\sqrt{2}}x - \frac{\sin 2x}{2\sqrt{2}} + C.$

(ii) $\int \frac{\cos 2x}{\sqrt{1+\cos 4x}}\,dx = \int \frac{\cos 2x}{\sqrt{2\cos^2 2x}}\,dx = \frac{1}{\sqrt{2}}\int dx = \frac{x}{\sqrt{2}} + C.$

EXAMPLE 12 *Evaluate:* *(i)* $\int \frac{dx}{a\sin x + b\cos x}$ *(ii)* $\int \frac{dx}{\sin x + \cos x}$

SOLUTION (i) Put $a = r\cos\theta$ and $b = r\sin\theta$ so that

$r^2 = (a^2 + b^2)$ and $\theta = \tan^{-1}(b/a).$

$$\therefore \int \frac{dx}{a\sin x + b\cos x} = \int \frac{dx}{r\cos\theta\sin x + r\sin\theta\cos x}$$

$$= \frac{1}{r}\int \frac{dx}{\sin(x+\theta)} = \frac{1}{r}\cdot\int \operatorname{cosec}(x+0)\,dx$$

$$= \frac{1}{r}\log\left\{\tan\left(\frac{0+x}{2}\right)\right\} + C$$

$$= \frac{1}{\sqrt{a^2+b^2}}\log\left[\tan\left\{\frac{1}{2}\tan^{-1}\left(\frac{b}{a}\right) + \frac{x}{2}\right\}\right] + C.$$

(ii) We can write,

$$\int \frac{dx}{\sin x + \cos x} = \frac{1}{\sqrt{2}}\int \frac{dx}{\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x}$$

$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\left(\cos\frac{\pi}{4}\sin x + \sin\frac{\pi}{4}\cos x\right)}$$

$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\sin\left(\frac{\pi}{4}+x\right)} = \frac{1}{\sqrt{2}}\int \operatorname{cosec}\left(\frac{\pi}{4}+x\right)dx$$

$$= \frac{1}{\sqrt{2}}\log\tan\left(\frac{\pi}{8}+\frac{x}{2}\right) + C.$$

EXAMPLE 13 *Evaluate:* (i) $\int \frac{dx}{4\cos x + 3\sin x}$ (ii) $\int \frac{dx}{(2\sin x + 3\cos x)^2}$

SOLUTION (i) Put $4 = r\sin 0$ and $3 = r\cos 0$ so that

$$r^2 = 25 \text{ and } 0 = \tan^{-1}\left(\frac{4}{3}\right).$$

$$\therefore \int \frac{dx}{4\cos x + 3\sin x} = \int \frac{dx}{r\sin\theta\cos x + r\cos\theta\sin x}$$

$$= \frac{1}{r}\int \frac{dx}{\sin(\theta+x)} = \frac{1}{r}\int \operatorname{cosec}(\theta+x)\,dx$$

$$= \frac{1}{r}\log\left\{\tan\left(\frac{0+x}{2}\right)\right\} + C$$

$$= \frac{1}{5}\log\left[\tan\left\{\frac{1}{2}\tan^{-1}\left(\frac{4}{3}\right) + \frac{x}{2}\right\}\right] + C.$$

(ii) Put $2 = r\cos\theta$ and $3 = r\sin\theta$ so that $r^2 = 13$ and $\theta = \tan^{-1}\left(\frac{3}{2}\right)$.

$$\because \int \frac{dx}{(2\sin x + 3\cos x)^2} = \int \frac{dx}{(r\cos\theta\sin x + r\sin\theta\cos x)^2}$$

$$= \frac{1}{r^2}\int \frac{dx}{\sin^2(0+x)} = \frac{1}{13}\cdot\int \operatorname{cosec}^2(0+x)\,dx$$

$$= -\frac{1}{13}\cot\,(0+x) + C$$

$$= -\frac{1}{13}\cot\left\{\tan^{-1}\left(\frac{3}{2}\right) + x\right\} + C.$$

EXAMPLE 14 *Evaluate* $\displaystyle\int \frac{\cos x}{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^3}dx.$

SOLUTION $\displaystyle\int \frac{\cos x}{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^3}dx = \int \frac{\cos^2(x/2) - \sin^2(x/2)}{\{\cos(x/2) + \sin(x/2)\}^3}dx$

$$= \int \frac{\cos(x/2) - \sin(x/2)}{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2}dx = 2\int \frac{1}{t^2}dt, \text{ where } t = \cos\frac{x}{2} + \sin\frac{x}{2}$$

$$= \frac{-2}{t} + C = \frac{-2}{\cos(x/2) + \sin(x/2)} + C.$$

EXAMPLE 15 *Evaluate* $\displaystyle\int \frac{dx}{\sqrt{1-\sin x}}.$

SOLUTION $\displaystyle\int \frac{dx}{\sqrt{1-\sin x}} = \int \frac{dx}{\left[\sin^2(x/2) + \cos^2(x/2) - 2\sin\frac{x}{2}\cos\frac{x}{2}\right]^{1/2}}$

$$= \int \frac{dx}{\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right)} = \frac{1}{\sqrt{2}}\int \frac{dx}{\left(\frac{1}{\sqrt{2}}\cdot\sin\frac{x}{2} - \cos\frac{x}{2}\cdot\frac{1}{\sqrt{2}}\right)}$$

$$= \frac{1}{\sqrt{2}}\cdot\int \frac{dx}{\left(\sin\frac{x}{2}\cos\frac{\pi}{4} - \cos\frac{x}{2}\sin\frac{\pi}{4}\right)}$$

$$= \frac{1}{\sqrt{2}}\int \operatorname{cosec}\left(\frac{x}{2} - \frac{\pi}{4}\right)dx = \frac{1}{\sqrt{2}}2\cdot\log\left[\tan\left(\frac{x}{4} - \frac{\pi}{8}\right)\right] + C$$

$$= \sqrt{2}\log\,\tan\left(\frac{x}{4} - \frac{\pi}{8}\right) + C.$$

EXAMPLE 16 *Evaluate* $\displaystyle\int \frac{\sin x}{\sqrt{1+\sin x}}dx.$

SOLUTION $\displaystyle\int \frac{\sin x}{\sqrt{1+\sin x}}dx = \int \frac{(1+\sin x) - 1}{\sqrt{1+\sin x}}dx$

$$= \int \sqrt{1+\sin x}\, dx - \int \frac{dx}{\sqrt{1+\sin x}}$$

$$= \int \sqrt{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}}\, dx$$

$$-\int \frac{dx}{\sqrt{\cos^2(x/2) + \sin^2(x/2) + 2\sin(x/2)\cos(x/2)}}$$

$$= \int [\cos(x/2) + \sin(x/2)]dx - \int \frac{dx}{[\cos(x/2) + \sin(x/2)]}$$

$$= \left(2\sin\frac{x}{2} - 2\cos\frac{x}{2}\right) - \frac{1}{\sqrt{2}} \cdot \int \frac{dx}{\frac{1}{\sqrt{2}}\cos\frac{x}{2} + \frac{1}{\sqrt{2}}\sin\frac{x}{2}}$$

$$= \left(2\sin\frac{x}{2} - 2\cos\frac{x}{2}\right) - \frac{1}{\sqrt{2}} \cdot \int \frac{dx}{\sin\left(\frac{x}{2} + \frac{\pi}{4}\right)}$$

$$= \left(2\sin\frac{x}{2} - 2\cos\frac{x}{2}\right) - \frac{1}{\sqrt{2}} \int \operatorname{cosec}\left(\frac{x}{2} + \frac{\pi}{4}\right) dx$$

$$= 2\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right) - \frac{1}{\sqrt{2}} \times 2\log\left|\tan\left(\frac{x}{4} + \frac{\pi}{8}\right)\right| + C$$

$$= 2\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right) - \sqrt{2}\log\left|\tan\left(\frac{x}{4} + \frac{\pi}{8}\right)\right| + C.$$

EXERCISE 13B

Evaluate the following integrals:

1. (i) $\int \sin^2 x\, dx$ (ii) $\int \cos^2 x\, dx$
2. (i) $\int \cos^2(x/2)\, dx$ (ii) $\int \cot^2(x/2)\, dx$
3. (i) $\int \sin^2 nx\, dx$ (ii) $\int \sin^5 x\, dx$
4. $\int \cos^3(3x+5)\, dx$ 5. $\int \sin^7(3-2x)\, dx$
6. (i) $\int \left(\frac{1-\cos 2x}{1+\cos 2x}\right) dx$ (ii) $\int \left(\frac{1+\cos 2x}{1-\cos 2x}\right) dx$
7. (i) $\int \left(\frac{1-\cos x}{1+\cos x}\right) dx$ (ii) $\int \left(\frac{1+\cos x}{1-\cos x}\right) dx$
8. $\int \sin 3x \cos 4x\, dx$ 9. $\int \cos 4x \cos 3x\, dx$ **[CBSE 2007]**
10. $\int \sin 4x \sin 8x\, dx$ **[CBSE 2007]** 11. $\int \sin 6x \cos x\, dx$

12. $\int \sin x\sqrt{1+\cos 2x}\,dx$
13. $\int \cos^4 x\,dx$ **[CBSE 2000, '03]**
14. $\int \cos 2x \cos 4x \cos 6x\,dx$
15. $\int \sin^3 x \cos x\,dx$
16. $\int \sec^4 x\,dx$
17. $\int \cos^3 x \sin^4 x\,dx$
18. $\int \cos^4 x \sin^3 x\,dx$
19. $\int \sin^{2/3} x \cos^3 x\,dx$
20. $\int \cos^{3/5} x \sin^3 x\,dx$
21. $\int \text{cosec}^4 2x\,dx$
22. $\int \frac{\cos 2x}{\cos x}dx$
23. $\int \frac{\cos x}{\cos(x+\alpha)}dx$
24. $\int \cos^3 x \sin 2x\,dx$
25. $\int \frac{\cos^9 x}{\sin x}dx$
26. $\int \cos^4 2x\,dx$
27. $\int \frac{\sin^2 x}{(1+\cos x)^2}dx$
28. $\int \frac{dx}{(3\cos x + 4\sin x)}$
29. $\int \frac{dx}{(a\cos x + b\sin x)^2}$, $a>0$ and $b>0$
30. $\int \frac{dx}{(\cos x - \sin x)}$
31. $\int (2\tan x - 3\cot x)^2 dx$
32. $\int \sin x \sin 2x \sin 3x\,dx$ **[CBSE 2012]**
33. $\int \left(\frac{1-\cot x}{1+\cot x}\right)dx$
34. $\int \frac{dx}{(2\sin x + \cos x + 3)}$ **[CBSE 2000C, '04]**

ANSWERS (EXERCISE 13B)

1. (i) $\frac{1}{2}x - \frac{1}{4}\sin 2x + C$ (ii) $\frac{1}{2}x + \frac{1}{4}\sin 2x + C$

2. (i) $\frac{1}{2}x + \frac{1}{2}\sin x + C$ (ii) $-2\cot(x/2) - x + C$

3. (i) $\frac{1}{2}x - \frac{\sin 2nx}{4n} + C$ (ii) $-\cos x - \frac{\cos^5 x}{5} + \frac{2\cos^3 x}{3} + C$

4. $\frac{1}{3}\sin(3x+5) - \frac{1}{9}\sin^3(3x+5) + C$

5. $\frac{1}{2}\cos(3-2x) - \frac{1}{14}\cos^7(3-2x) - \frac{1}{2}\cos^3(3-2x) + \frac{3}{10}\cos^5(3-2x) + C$

6. (i) $\tan x - x + C$ (ii) $-\cot x - x + C$

7. (i) $2\tan\frac{x}{2} - x + C$ (ii) $-2\cot\frac{x}{2} - x + C$

8. $\frac{-\cos 7x}{14} + \frac{\cos x}{2} + C$ **9.** $\frac{\sin 7x}{14} + \frac{1}{2}\sin x + C$

10. $\frac{\sin 4x}{8} - \frac{\sin 12x}{24} + C$ **11.** $\frac{-\cos 7x}{14} - \frac{\cos 5x}{10} + C$

12. $\frac{\sin^2 x}{\sqrt{2}} + C$ **13.** $\frac{3x}{8} + \frac{\sin 4x}{32} + \frac{\sin 2x}{4} + C$

14. $\frac{x}{4} + \frac{\sin 4x}{16} + \frac{\sin 8x}{32} + \frac{\sin 12x}{48} + C$ **15.** $\frac{\sin^4 x}{4} + C$

16. $\tan x + \frac{1}{3}\tan^3 x + C$ **17.** $\frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + C$

18. $\frac{1}{7}\cos^7 x - \frac{1}{5}\cos^5 x + C$ **19.** $\frac{3}{5}\sin^{5/3} x - \frac{3}{11}\sin^{11/3} x + C$

20. $-\frac{5}{8}\cos^{8/5} x + \frac{5}{18}\cos^{18/5} x + C$ **21.** $-\frac{1}{2}\cot 2x - \frac{1}{6}\cot^3 2x + C$

22. $2\sin x - \log|\sec x + \tan x| + C$ **23.** $x\cos\alpha - \sin\alpha \cdot \log|\cos(x+\alpha)| + C$

24. $-\frac{2}{5}\cos^5 x + C$ **25.** $\log|\sin x| - 2\sin^2 x + \frac{3}{2}\sin^4 x - \frac{2}{3}\sin^6 x + \frac{1}{8}\sin^8 x + C$

26. $\frac{3x}{8} + \frac{\sin 4x}{8} + \frac{\sin 8x}{64} + C$ **27.** $2\tan\frac{x}{2} - x + C$

28. $\frac{1}{5}\log\tan\left[\frac{1}{2}\tan^{-1}\frac{3}{4} + \frac{x}{2}\right] + C$ **29.** $\frac{1}{(a^2+b^2)}\tan\left\{x - \tan^{-1}\frac{b}{a}\right\} + C$

30. $\frac{-1}{\sqrt{2}}\log\left|\tan\left(\frac{\pi}{8} - \frac{x}{2}\right)\right| + C$ **31.** $4\tan x - 9\cot x - 25x + C$

32. $\frac{1}{24}\cos 6x - \frac{1}{16}\cos 4x - \frac{1}{8}\cos 2x + C$

33. $-\log|\sin x + \cos x| + C$ **34.** $\tan^{-1}\left(1 + \tan\frac{x}{2}\right) + C$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 13B)

12. $I = \int \sin x \cdot \sqrt{2\cos^2 x}\, dx = \sqrt{2}\int \sin x \cos x\, dx$. Put $\sin x = t$.

13. $I = \frac{1}{4}\int (2\cos^2 x)^2\, dx = \frac{1}{4}\int (1+\cos 2x)^2\, dx$

$= \frac{1}{4}\int\left[(1+2\cos 2x) + \left(\frac{1+\cos 4x)}{2}\right)\right] dx = \int\left[\frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x\right] dx.$

16. $I = \int (1+\tan^2 x)\sec^2 x\, dx$. Put $\tan x = t$.

17. $I = \int \cos x(1-\sin^2 x)\sin^4 x\, dx$. Put $\sin x = t$.

19. $I = \int \sin^{2/3} x(1-\sin^2 x)\cos x\, dx$. Put $\sin x = t$.

21. $I = \int (1+\cot^2 2x)\operatorname{cosec}^2 2x\, dx$. Put $\cot 2x = t$.

22. $I = \int \frac{(2\cos^2 x - 1)}{\cos x} dx = 2\int \cos x\, dx - \int \sec x\, dx.$

24. $I = 2\int \cos^4 x \sin x\, dx$. Put $\cos x = t$.

25. $I = \int \dfrac{\cos^8 x \cos x}{\sin x} dx = \int \dfrac{(1-\sin^2 x)^4 \cos x}{\sin x} dx$. Put $\sin x = t$.

27. $I = \int \dfrac{(1-\cos^2 x)}{(1+\cos x)^2} dx = \int \dfrac{(1-\cos x)}{(1+\cos x)} dx = \int \tan^2 \dfrac{x}{2} dx = \int \left(\sec^2 \dfrac{x}{2} - 1 \right) dx.$

28. Put $3 = r \sin\theta$ and $4 = r \cos\theta$.

29. Put $a = r \sin\theta$ and $b = r \cos\theta$.

30. $I = \dfrac{1}{\sqrt{2}} \cdot \int \dfrac{dx}{\left(\dfrac{1}{\sqrt{2}} \cos x - \dfrac{1}{\sqrt{2}} \sin x \right)} = \dfrac{1}{\sqrt{2}} \int \dfrac{dx}{\sin\left(\dfrac{\pi}{4} - x \right)} = \dfrac{1}{\sqrt{2}} \cdot \int \operatorname{cosec}\left(\dfrac{\pi}{4} - x \right) dx.$

32. $I = \dfrac{1}{4} \int (\sin 2x + \sin 4x - \sin 6x)\, dx.$

33. $I = -\int \dfrac{(\cos x - \sin x)}{(\sin x + \cos x)} dx$. Put $(\sin x + \cos x) = t$.

34. $\tan \dfrac{x}{2} = t$, $dx = \dfrac{2dt}{(1+t^2)}$; $\sin x = \dfrac{2t}{(1+t^2)}$ and $\cos x = \dfrac{1-t^2}{1+t^2}$.

Integration by Parts

THEOREM *If u and v are two functions of x then*

$$\int (uv)\, dx = [u \cdot \int v\, dx] - \int \left\{ \frac{du}{dx} \cdot \int v\, dx \right\} dx.$$

PROOF For any two functions $f_1(x)$ and $f_2(x)$, we have

$$\frac{d}{dx}[f_1(x) \cdot f_2(x)] = f_1(x) \cdot f_2'(x) + f_2(x) \cdot f_1'(x).$$

$\therefore$ $\int \{f_1(x) \cdot f_2'(x) + f_2(x) \cdot f_1'(x)\}\, dx = f_1(x) \cdot f_2(x)$

or $\int f_1(x) \cdot f_2'(x)\, dx + \int f_2(x) \cdot f_1'(x)\, dx = f_1(x) \cdot f_2(x)$

or $\int f_1(x) \cdot f_2'(x)\, dx = f_1(x) \cdot f_2(x) - \int f_2(x) \cdot f_1'(x)\, dx.$

Let $f_1(x) = u$ and $f_2'(x) = v$ so that $f_2(x) = \int v\, dx$.

$\therefore$ $\int (uv)\, dx = u \cdot \int v\, dx - \int \left\{ \dfrac{du}{dx} \cdot \int v\, dx \right\} dx.$

We can express this result as given below:

Integral of product of two functions

$$= \textbf{(1st function)} \times \textbf{(integral of 2nd)} - \int \textbf{\{(derivative of 1st)} \times \textbf{(integral of 2nd)\}}\, dx.$$

REMARKS (i) If the integrand is of the form $f(x)\,x^n$, we consider x^n as the first function and $f(x)$ as the second function.

(ii) If the integrand contains a *logarithmic* or an *inverse trigonometric* function, we take it as the first function. In all such cases, if the second function is not given, we take it as 1.

SOLVED EXAMPLES

EXAMPLE 1 *Evaluate:*

(i) $\int x \sec^2 x\,dx$ *(ii)* $\int x \sin 2x\,dx$

SOLUTION (i) Integrating by parts, taking x as the first function, we get

$$\int x \sec^2 x\,dx = x \cdot \int \sec^2 x\,dx - \int \left\{ \frac{d}{dx}(x) \cdot \int \sec^2 x\,dx \right\} dx$$

$$= x \tan x - \int 1 \cdot \tan x\,dx = x \tan x + \log |\cos x| + C.$$

(ii) Integrating by parts, taking x as the first function, we get

$$\int x \sin 2x\,dx = x \cdot \int \sin 2x\,dx - \int \left\{ \frac{d}{dx}(x) \cdot \int \sin 2x\,dx \right\} dx$$

$$= x \cdot \left(\frac{-\cos 2x}{2} \right) - \int 1 \cdot \left(\frac{-\cos 2x}{2} \right) dx$$

$$= \frac{-x \cos 2x}{2} + \frac{1}{2} \int \cos 2x\,dx$$

$$= \frac{-x \cos 2x}{2} + \frac{1}{2} \cdot \frac{\sin 2x}{2} + C$$

$$= \frac{-x \cos 2x}{2} + \frac{1}{4} \sin 2x + C.$$

EXAMPLE 2 *Evaluate* $\int x^n \log x\,dx.$

SOLUTION Integrating by parts, taking $\log x$ as the first function, we get

$$\int x^n \log x = (\log x) \cdot \int x^n dx - \int \left\{ \frac{d}{dx}(\log x) \cdot \int x^n dx \right\} dx$$

$$= (\log x) \cdot \frac{x^{n+1}}{(n+1)} - \int \frac{1}{x} \cdot \frac{x^{n+1}}{(n+1)} dx$$

$$= \frac{x^{n+1} \log x}{(n+1)} - \frac{1}{(n+1)} \int x^n dx$$

$$= \frac{x^{n+1} \log x}{(n+1)} - \frac{x^{n+1}}{(n+1)^2} + C.$$

EXAMPLE 3 *Evaluate* $\int x^2 \sin x \, dx$.

SOLUTION Integrating by parts, taking x^2 as the first function, we get

$$\int x^2 \sin x \, dx = x^2 \int \sin x \, dx - \int \left[\frac{d}{dx}(x^2) \cdot \int \sin x \, dx \right] dx$$

$$= x^2(-\cos x) - \int 2x(-\cos x) \, dx$$

$$= -x^2 \cos x + 2\int x \cos x \, dx$$

$$= -x^2 \cos x + 2\left[x(\sin x) - \int \left\{ \frac{d}{dx}(x) \cdot \int \cos x \, dx \right\} dx \right]$$

[integrating $x \cos x$ by parts]

$$= -x^2 \cos x + 2[x \sin x - \int \sin x \, dx]$$

$$= -x^2 \cos x + 2[x \sin x + \cos x] + C.$$

EXAMPLE 4 *Evaluate* $\int x \cos^2 x \, dx$.

SOLUTION $$\int x \cos^2 x \, dx = \int x \left(\frac{1 + \cos 2x}{2} \right) dx = \frac{1}{2} \int x \, dx + \frac{1}{2} \int x \cos 2x \, dx$$

$$= \frac{x^2}{4} + \frac{1}{2} \cdot \left[x \cdot \int \cos 2x \, dx - \int \left\{ \frac{d}{dx}(x) \cdot \int \cos 2x \, dx \right\} dx \right]$$

[integrating $x \cos 2x$ by parts]

$$= \frac{x^2}{4} + \frac{1}{2} \left[\frac{x \sin 2x}{2} - \int \frac{\sin 2x}{2} dx \right]$$

$$= \frac{x^2}{4} + \frac{x \sin 2x}{4} - \frac{1}{4} \cdot \frac{(-\cos 2x)}{2} + C$$

$$= \frac{x^2}{4} + \frac{x \sin 2x}{4} + \frac{\cos 2x}{8} + C.$$

EXAMPLE 5 *Evaluate* $\int \log x \, dx$.

SOLUTION Integrating by parts, taking $\log x$ as the first function and 1 as the second function, we get

$$\int \log x \, dx = \int (\log x \cdot 1) \, dx$$

$$= (\log x) \cdot \int 1 \, dx - \int \left\{ \frac{d}{dx}(\log x) \cdot \int 1 \, dx \right\} dx$$

$$= (\log x) \cdot x - \int \left(\frac{1}{x} \cdot x \right) dx = x \log x - \int dx$$

$$= x \log x - x + C = x(\log x - 1) + C.$$

EXAMPLE 6 *Evaluate* $\int \log (1 + x^2) \, dx$.

SOLUTION Integrating by parts, taking $\log (1 + x^2)$ as the first function and 1

as the second function, we get

$$\begin{aligned}\int \log(1+x^2)\,dx &= \int \{\log(1+x^2)\cdot 1\}\,dx \\ &= \log(1+x^2)\cdot\int 1\,dx - \int\left[\frac{d}{dx}\{\log(1+x^2)\}\cdot\int 1\,dx\right]dx \\ &= \log(1+x^2)\cdot x - \int\frac{2x}{(1+x^2)}\cdot x\,dx \\ &= x\log(1+x^2) - 2\int\frac{x^2}{(1+x^2)}\,dx \\ &= x\log(1+x^2) - 2\int\left(1-\frac{1}{1+x^2}\right)dx \\ &= x\log(1+x^2) - 2\int dx + 2\int\frac{dx}{(1+x^2)} \\ &= x\log(1+x^2) - 2x + 2\tan^{-1}x + C.\end{aligned}$$

EXAMPLE 7 *Evaluate* $\int(\log x)^2\,dx$.

SOLUTION Integrating by parts, taking $(\log x)^2$ as the first function and 1 as the second function, we get

$$\begin{aligned}\int(\log x)^2\,dx &= \int\{(\log x)^2\cdot 1\}\,dx \\ &= (\log x)^2\cdot\int 1\,dx - \int\left\{\frac{d}{dx}(\log x)^2\cdot\int 1\,dx\right\}dx \\ &= x(\log x)^2 - \int\left(\frac{2\log x}{x}\cdot x\right)dx \\ &= x(\log x)^2 - 2\int(\log x\cdot 1)\,dx \\ &= x(\log x)^2 - 2\left[(\log x)\int dx - \int\left\{\frac{d}{dx}(\log x)\cdot\int dx\right\}dx\right] \\ &= x(\log x)^2 - 2\left[x\log x - \int\frac{1}{x}\cdot x\,dx\right] \\ &= x(\log x)^2 - 2x\log x + 2x + C.\end{aligned}$$

EXAMPLE 8 *Evaluate* $\int\frac{\log x}{x^2}dx$.

SOLUTION Integrating by parts, taking $\log x$ as the first function and $\frac{1}{x^2}$ as the second function, we get

$$\begin{aligned}\int\frac{\log x}{x^2}\,dx &= \int(\log x)\cdot\frac{1}{x^2}\,dx \\ &= (\log x)\cdot\int\frac{1}{x^2}dx - \int\left\{\frac{d}{dx}(\log x)\cdot\int\frac{1}{x^2}dx\right\}dx \\ &= (\log x)\left(-\frac{1}{x}\right) - \int\frac{1}{x}\cdot\left(-\frac{1}{x}\right)dx = -\frac{\log x}{x} + \int\frac{1}{x^2}\,dx \\ &= -\frac{\log x}{x} - \frac{1}{x} + C = \frac{-(\log x+1)}{x} + C.\end{aligned}$$

EXAMPLE 9 *Evaluate* $\int e^{2x} \sin x \, dx.$

SOLUTION Integrating by parts, we get

$$\int e^{2x} \sin x \, dx = (e^{2x} \cdot \int \sin x \, dx) - \int \left\{ \frac{d}{dx}(e^{2x}) \cdot \int \sin x \, dx \right\} dx$$

$$= e^{2x} \cdot (-\cos x) - 2\int e^{2x}(-\cos x) \, dx$$

$$= -e^{2x} \cos x + 2 \int e^{2x} \cos x \, dx$$

$$= -e^{2x} \cos x + 2\left[(e^{2x} \cdot \int \cos x \, dx) - \int \left\{ \frac{d}{dx}(e^{2x}) \cdot \int \cos x \, dx \right\} dx \right]$$

[integrating $e^{2x} \cos x$ by parts]

$$= -e^{2x} \cos x + 2e^{2x} \sin x - 4\int e^{2x} \sin x \, dx + C.$$

$$\therefore \quad 5\int e^{2x} \sin x \, dx = e^{2x}(2\sin x - \cos x) + C$$

$$\text{or} \quad \int e^{2x} \sin x \, dx = \frac{1}{5} e^{2x}(2\sin x - \cos x) + C.$$

EXAMPLE 10 *Evaluate* $\int \left(\frac{x - \sin x}{1 - \cos x} \right) dx.$

SOLUTION $$\int \left(\frac{x - \sin x}{1 - \cos x} \right) dx = \int \frac{x}{(1 - \cos x)} dx - \int \frac{\sin x}{(1 - \cos x)} dx$$

$$= \int \frac{x}{2\sin^2(x/2)} dx - \int \frac{2\sin(x/2)\cos(x/2)}{2\sin^2(x/2)} dx$$

$$= \frac{1}{2}\int x \operatorname{cosec}^2(x/2) \, dx - \int \cot(x/2) \, dx$$

$$= \frac{1}{2}\left[x \cdot \int \operatorname{cosec}^2(x/2) dx - \int \left\{ \frac{d}{dx}(x) \cdot \int \operatorname{cosec}^2(x/2) dx \right\} dx \right]$$

$$- \int \cot(x/2) \, dx \quad \text{[integrating by parts]}$$

$$= \frac{1}{2}\left[x \cdot \left(-2\cot \frac{x}{2} \right) - \int \left[1 \cdot \left(-2\cot \frac{x}{2} \right) \right] dx \right] - \int \cot(x/2) dx + C$$

$$= -x \cot \frac{x}{2} + \int \cot \frac{x}{2} dx - \int \cot \frac{x}{2} dx + C = -x \cot \frac{x}{2} + C.$$

EXAMPLE 11 *Evaluate* $\int x \tan^{-1} x \, dx.$ **[CBSE 2000C, '04C]**

SOLUTION Integrating by parts, taking $\tan^{-1} x$ as the first function, we get

$$\int x \tan^{-1} x \, dx = (\tan^{-1} x) \cdot \int x \, dx - \int \left\{ \frac{d}{dx}(\tan^{-1} x) \cdot \int x \, dx \right\} dx$$

$$= (\tan^{-1} x) \cdot \frac{x^2}{2} - \int \frac{1}{(1 + x^2)} \cdot \frac{x^2}{2} dx$$

$$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2}\int \left(1 - \frac{1}{1 + x^2} \right) dx \quad \text{[on dividing } x^2 \text{ by } 1 + x^2\text{]}$$

$$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2}\int dx + \frac{1}{2}\int \frac{1}{(1+x^2)}\,dx$$

$$= \frac{x^2 \tan^{-1} x}{2} - \frac{x}{2} + \frac{1}{2}\tan^{-1} x + C$$

$$= \frac{1}{2}(1+x^2)\tan^{-1} x - \frac{1}{2}x + C.$$

EXAMPLE 12 *Evaluate* $\int x^2 \sin^{-1} x\,dx$.

SOLUTION Integrating by parts, taking $\sin^{-1} x$ as the first function we get

$$\int x^2 \sin^{-1} x\,dx = (\sin^{-1} x)\cdot\frac{x^3}{3} - \int \frac{1}{\sqrt{1-x^2}}\cdot\frac{x^3}{3}\,dx$$

$$= \frac{x^3 \sin^{-1} x}{3} - \frac{1}{3}\int \frac{x^3}{\sqrt{1-x^2}}\,dx$$

$$= \frac{x^3 \sin^{-1} x}{3} - \frac{1}{3}\int \frac{x\cdot x^2}{\sqrt{1-x^2}}\,dx$$

$$= \frac{x^3 \sin^{-1} x}{3} + \frac{1}{3}\int \frac{t(1-t^2)}{t}\,dt, \text{ where } (1-x^2) = t^2$$

$$= \frac{x^3 \sin^{-1} x}{3} + \frac{1}{3}\int dt - \frac{1}{3}\int t^2 dt$$

$$= \frac{x^3 \sin^{-1} x}{3} + \frac{1}{3}t - \frac{1}{9}t^3 + C$$

$$= \frac{x^3 \sin^{-1} x}{3} + \frac{1}{3}\sqrt{1-x^2} - \frac{1}{9}(1-x^2)^{3/2} + C.$$

EXAMPLE 13 *Evaluate:*

(i) $\int \cos^{-1} x\,dx$ *(ii)* $\int \tan^{-1} x\,dx$ *(iii)* $\int \sec^{-1} x\,dx$

SOLUTION (i) Put $\cos^{-1} x = t$ so that $x = \cos t$ and $dx = -\sin t\,dt$.

$$\therefore \quad \int \cos^{-1} x\,dx = -\int t \sin t\,dt$$

$$= -[t\cdot(-\cos t) - \int 1\cdot(-\cos t)\,dt]$$

[integrating by parts]

$$= t\cos t - \int \cos t\,dt = t\cos t - \sin t + C$$

$$= x\cos^{-1} x - \sqrt{1-x^2} + C$$

$[\because \quad \cos t = x \Rightarrow \sin t = \sqrt{1-x^2}\,]$.

(ii) Put $\tan^{-1} x = t$ so that $x = \tan t$ and $dx = \sec^2 t\,dt$.

$$\therefore \quad \int \tan^{-1} x\,dx = \int t\sec^2 t\,dt$$

$$= t\cdot\tan t - \int 1\cdot\tan t\,dt \qquad \text{[integrating by parts]}$$

$$= t\cdot\tan t + \log|\cos t| + C$$

$$= (\tan^{-1}x)\cdot x + \log\left|\frac{1}{\sqrt{1+x^2}}\right| + C$$

$$\left[\because \tan t = x \Rightarrow \cos t = \frac{1}{\sqrt{1+x^2}}\right]$$

$$= x(\tan^{-1}x) - \frac{1}{2}\log|1+x^2| + C.$$

(iii) Put $\sec^{-1}x = t$ so that $x = \sec t$ and $dx = \sec t \tan t\, dt$.

$$\therefore \int \sec^{-1}x\, dx = \int t(\sec t \tan t)\, dt$$

$$= t(\sec t) - \int 1\cdot \sec t\, dt \quad \text{[integrating by parts]}$$

$$= t(\sec t) - \log|\sec t + \tan t| + C$$

$$= t(\sec t) - \log|\sec t + \sqrt{\sec^2 t - 1}| + C$$

$$= x(\sec^{-1}x) - \log|x + \sqrt{x^2-1}| + C.$$

EXAMPLE 14 *Evaluate* $\int (\sin^{-1}x)^2 dx$. **[CBSE 2004]**

SOLUTION Putting $x = \sin t$ and $dx = \cos t\, dt$, we get

$$\int (\sin^{-1}x)^2 dx = \int t^2 \cos t\, dt$$

$$= t^2\cdot(\sin t) - \int 2t(\sin t)\, dt \quad \text{[integrating by parts]}$$

$$= t^2 \sin t - 2[t(-\cos t) - \int 1\cdot(-\cos t)\, dt]$$

[integrating $t(\sin t)$ by parts]

$$= t^2 \sin t + 2t\cos t - 2\sin t + C$$

$$= x(\sin^{-1}x)^2 + 2(\sin^{-1}x)\sqrt{1-x^2} - 2x + C.$$

EXAMPLE 15 *Evaluate* $\int \frac{\sin^{-1}x}{(1-x^2)^{3/2}} dx$.

SOLUTION Put $x = \sin t$ so that $dx = \cos t\, dt$ and $t = \sin^{-1}x$.

$$\therefore \int \frac{\sin^{-1}x}{(1-x^2)^{3/2}} dx = \int \frac{t\cos t}{(1-\sin^2 t)^{3/2}} dt = \int \frac{t\cos t}{\cos^3 t} dt$$

$$= \int t\sec^2 t\, dt$$

$$= t\cdot(\tan t) - \int 1\cdot \tan t\, dt \quad \text{[integrating by parts]}$$

$$= t\cdot(\tan t) + \log|\cos t| + C$$

$$= (\sin^{-1}x)\cdot\frac{x}{\sqrt{1-x^2}} + \log|\sqrt{1-x^2}| + C.$$

$$\left[\because \cos t = \sqrt{1-x^2} \text{ and } \tan t = \frac{x}{\sqrt{1-x^2}}\right]$$

$$= \frac{x(\sin^{-1}x)}{\sqrt{1-x^2}} + \frac{1}{2}\log|(1-x^2)| + C.$$

EXAMPLE 16 *Evaluate* $\int \frac{x \tan^{-1} x}{(1+x^2)^{3/2}} dx.$

SOLUTION Put $x = \tan t$ so that $dx = \sec^2 t\, dt.$

$$\therefore \int \frac{x \tan^{-1} x}{(1+x^2)^{3/2}} dx = \int \frac{(\tan t)\, t}{(1+\tan^2 t)^{3/2}} \cdot \sec^2 t\, dt$$

$$= \int \frac{(\tan t) t}{\sec t} dt = \int t \sin t\, dt$$

$$= t(-\cos t) - \int 1 \cdot (-\cos t)\, dt \quad \text{[integrating by parts]}$$

$$= -t \cos t + \sin t + C = \frac{-\tan^{-1} x}{\sqrt{1+x^2}} + \frac{x}{\sqrt{1+x^2}} + C$$

$$\left[\because \quad \sin t = \frac{x}{\sqrt{1+x^2}} \quad \text{and} \quad \cos t = \frac{1}{\sqrt{1+x^2}} \right].$$

EXAMPLE 17 *Evaluate:*

(i) $\int \sin^{-1}(3x - 4x^3)\, dx$ *(ii)* $\int \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx$ **[CBSE 2002]**

(iii) $\int \tan^{-1} \sqrt{\frac{1-x}{1+x}}\, dx$ *(iv)* $\int \sin^{-1} \sqrt{\frac{x}{a+x}}\, dx$

SOLUTION (i) Put $x = \sin t$ so that $dx = \cos t\, dt.$

$$\therefore \int \sin^{-1}(3x - 4x^3)\, dx = \int \sin^{-1}(3 \sin t - 4 \sin^3 t) \cos t\, dt$$

$$= \int \sin^{-1}(\sin 3t) \cos t\, dt$$

$$= 3 \int t \cos t\, dt$$

$$= 3[t(\sin t) - \int 1 \cdot \sin t\, dt]$$

[integrating by parts]

$$= 3t \sin t + 3 \cos t + C$$

$$= 3x(\sin^{-1} x) + 3\sqrt{1-x^2} + C.$$

(ii) Put $x = \tan t$ so that $dx = \sec^2 t\, dt.$

$$\therefore \int \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx = \int \sin^{-1}\left(\frac{2 \tan t}{1+\tan^2 t}\right) \sec^2 t\, dt$$

$$= \int \sin^{-1}(\sin 2t) \sec^2 t\, dt = 2 \int t \cdot \sec^2 t\, dt$$

$$= 2[t \cdot \tan t - \int 1 \cdot \tan t\, dt]$$

$$= 2t \cdot \tan t + 2 \log |\cos t| + C$$

$$= 2x(\tan^{-1} x) + 2 \log \left| \frac{1}{\sqrt{1+x^2}} \right| + C$$

$$= 2x(\tan^{-1} x) + 2 \cdot \left(-\frac{1}{2}\right) \log |1 + x^2| + C$$

$$= 2x(\tan^{-1} x) - \log |1 + x^2| + C.$$

(iii) Put $x = \cos t$ so that $dx = -\sin t\, dt$.

$$\therefore \quad \int \tan^{-1}\sqrt{\frac{1-x}{1+x}}\,dx = \int \tan^{-1}\sqrt{\frac{1-\cos t}{1+\cos t}}(-\sin t)\,dt$$

$$= \int \tan^{-1}\sqrt{\frac{2\sin^2(t/2)}{2\cos^2(t/2)}}(-\sin t)\,dt$$

$$= \int\left[\tan^{-1}\left(\tan\frac{t}{2}\right)\right](-\sin t)\,dt = -\frac{1}{2}\int t\,(\sin t)\,dt$$

$$= -\frac{1}{2}[t(-\cos t) - \int 1\cdot(-\cos t)\,dt] \quad \text{[integrating by parts]}$$

$$= \frac{1}{2}t\cdot\cos t - \frac{1}{2}\sin t + C = \frac{1}{2}x(\cos^{-1}x) - \frac{1}{2}\sqrt{1-x^2} + C.$$

(iv) Put $x = a\tan^2 t$ so that $dx = (2a\sec^2 t\tan t)\,dt$.

$$\therefore \quad \int \sin^{-1}\sqrt{\frac{x}{a+x}}\,dx = \int \sin^{-1}\left\{\sqrt{\frac{a\tan^2 t}{a(1+\tan^2 t)}}\right\}2a\sec^2 t\tan t\,dt$$

$$= 2a\int t(\sec^2 t\cdot\tan t)\,dt$$

$$= 2a\left[t\cdot\frac{1}{2}\tan^2 t - \int 1\cdot\frac{1}{2}\tan^2 t\,dt\right]$$

[integrating by parts and using $\int \sec^2 t\tan t\,dt = \frac{1}{2}\tan^2 t$]

$$= at(\tan^2 t) - a\int(\sec^2 t - 1)\,dt$$

$$= at(\tan^2 t) - a\int \sec^2 t\,dt + a\int dt$$

$$= at(\tan^2 t) - a\tan t + at + C$$

$$= a\left(\tan^{-1}\sqrt{\frac{x}{a}}\right)\cdot\left(\frac{x}{a}\right) - a\cdot\sqrt{\frac{x}{a}} + a\tan^{-1}\sqrt{\frac{x}{a}} + C$$

$$= x\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{ax} + a\tan^{-1}\sqrt{\frac{x}{a}} + C.$$

EXAMPLE 18 *Evaluate* $\int x\cos^3 x\sin x\,dx$.

SOLUTION Take x as the first function and $(\cos^3 x\sin x)$ as the second.

Putting $\cos x = t$, we can evaluate $\int \cos^3 x\sin x\,dx$ as $-\frac{1}{4}\cos^4 x$.

So, integrating by parts, we get

$$\int x\cos^3 x\sin x\,dx = x\cdot\left(\frac{-1}{4}\cos^4 x\right) - \int 1\cdot\left(-\frac{1}{4}\right)\cos^4 x\,dx$$

$$= -\frac{x}{4}\cos^4 x + \frac{1}{4}\int\left(\frac{1+\cos 2x}{2}\right)^2 dx$$

$$= -\frac{x\cos^4 x}{4} + \frac{1}{4}\int\left(\frac{1}{4} + \frac{\cos^2 2x}{4} + \cos 2x\right)dx$$

$$= -\frac{x\cos^4 x}{4} + \frac{1}{16}\int dx + \frac{1}{4}\int \cos 2x\, dx + \frac{1}{32}\int 2\cos^2 2x\, dx$$

$$= -\frac{x\cos^4 x}{4} + \frac{x}{16} + \frac{\sin 2x}{8} + \frac{1}{32}\int (1 + \cos 4x)\, dx + C$$

$$= -\frac{x\cos^4 x}{4} + \frac{x}{16} + \frac{\sin 2x}{8} + \frac{1}{32}\int dx + \frac{1}{32}\int \cos 4x\, dx$$

$$= -\frac{x\cos^4 x}{4} + \frac{3x}{32} + \frac{\sin 2x}{8} + \frac{\sin 4x}{128} + C.$$

EXAMPLE 19 *Evaluate* $\int \sin(\log x)\, dx$.

SOLUTION Put $\log x = t$ so that $x = e^t$ and $\frac{1}{x}dx = dt$ or $dx = e^t dt$.

$$\therefore \quad \int \sin(\log x)\, dx = \int e^t \sin t\, dt \qquad \ldots \text{(i)}$$

Now, $\int e^t \sin t\, dt = e^t(-\cos t) - \int e^t \cdot(-\cos t)\, dt$ [integrating by parts]

$$= -e^t \cos t + \int e^t \cos t\, dt$$

$$= -e^t \cos t + [e^t \sin t - \int e^t \sin t\, dt]$$

[integrating $e^t \cos t$ by parts]

$$= -e^t \cos t + e^t \sin t - \int e^t \sin t\, dt.$$

$$\therefore \quad 2\int e^t \sin t\, dt = -e^t \cos t + e^t \sin t$$

$$\text{or} \quad \int e^t \sin t\, dt = \frac{1}{2}(-e^t \cos t + e^t \sin t) + C.$$

Putting this value in (i), we get

$$\int \sin(\log x)\, dx = \int e^t \sin t\, dt$$

$$= \frac{1}{2}(-e^t \cos t + e^t \sin t) + C$$

$$= \frac{1}{2}[-x\cos(\log x) + x\sin(\log x)] + C$$

$$= -\frac{1}{2}x\cos(\log x) + \frac{1}{2}x\sin(\log x) + C.$$

EXAMPLE 20 *Evaluate* $\int \sin\sqrt{x}\, dx$.

SOLUTION Put $\sqrt{x} = t$ so that $\frac{1}{2\sqrt{x}}dx = dt$ or $dx = 2t\, dt$.

$$\therefore \quad \int \sin\sqrt{x}\, dx = 2\int t \sin t\, dt = 2[t(-\cos t) - \int 1\cdot(-\cos t)\, dt]$$

[integrating $t \sin t$ by parts]

$$= -2t\cos t + 2\sin t + C$$
$$= -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C.$$

EXAMPLE 21 *Evaluate* $\int \sec^3 x\,dx$. **[CBSE 2003]**

SOLUTION $\int \sec^3 x\,dx = \int \sec x \cdot \sec^2 x\,dx$

$$= \sec x \cdot (\tan x) - \int \sec x \tan x(\tan x)\,dx$$

[integrating by parts]

$$= \sec x \tan x - \int \sec x(\sec^2 x - 1)\,dx$$
$$= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx$$

$$\therefore \quad 2\int \sec^3 x\,dx = \sec x \tan x + \log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + C$$

$$\text{or} \quad \int \sec^3 x\,dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + C'.$$

EXAMPLE 22 *Evaluate* $\int \tan^{-1}\sqrt{x}\,dx$.

SOLUTION Put $\sqrt{x} = t$ so that $\frac{1}{2\sqrt{x}}dx = dt$ or $dx = 2t\,dt$.

$$\therefore \quad \int \tan^{-1}\sqrt{x}\,dx = 2\int t(\tan^{-1}t)\,dt$$
$$= 2\left[(\tan^{-1}t)\cdot\frac{t^2}{2} - \int\left\{\frac{1}{(1+t^2)}\cdot\frac{t^2}{2}\right\}dt\right] + C$$
$$= t^2(\tan^{-1}t) - \int\frac{t^2}{(1+t^2)}dt + C$$
$$= t^2(\tan^{-1}t) - \int\frac{[(1+t^2)-1]}{(1+t^2)}dt + C$$
$$= t^2(\tan^{-1}t) - \int dt + \int\frac{1}{(1+t^2)}dt + C$$
$$= t^2(\tan^{-1}t) - t + \tan^{-1}t + C = (t^2+1)\tan^{-1}t - t + C$$
$$= (x+1)\tan^{-1}\sqrt{x} - \sqrt{x} + C.$$

EXAMPLE 23 *Evaluate* $\int \frac{\tan^{-1}x}{(1+x)^2}dx$. **[CBSE 2003]**

SOLUTION Integrating by parts, taking $\tan^{-1}x$ as the first function and $\frac{1}{(1+x)^2}$ as the second function, we get

$$I = \tan^{-1}x\cdot\frac{(-1)}{(1+x)} - \int\frac{1}{(1+x^2)}\cdot\frac{(-1)}{(1+x)}dx$$
$$= \frac{-\tan^{-1}x}{(1+x)} + \int\frac{dx}{(1+x)(1+x^2)} = \frac{-\tan^{-1}x}{(1+x)} + \frac{1}{2}\cdot\int\left\{\frac{1}{(1+x)} + \frac{(1-x)}{(1+x^2)}\right\}$$

[by partial fractions]

$$= \frac{-\tan^{-1}x}{(1+x)} + \frac{1}{2}\log|1+x| + \frac{1}{2}\tan^{-1}x - \frac{1}{4}\log(1+x^2) + C.$$

EXAMPLE 24 *Evaluate* $\int\left\{\frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}\right\}dx.$ **[CBSE 2014C]**

SOLUTION We have

$$I = \int \frac{\left\{\sin^{-1}\sqrt{x} - \left(\frac{\pi}{2} - \sin^{-1}\sqrt{x}\right)\right\}}{\left(\frac{\pi}{2}\right)}dx \quad [\because \sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x} = \frac{\pi}{2}]$$

$$= \frac{2}{\pi}\int\left(2\sin^{-1}\sqrt{x} - \frac{\pi}{2}\right)dx$$

$$= \frac{4}{\pi}\int \sin^{-1}\sqrt{x}\,dx - \int dx = \frac{4}{\pi}\int \sin^{-1}\sqrt{x}\,dx - x + C. \quad \ldots \text{(i)}$$

Putting $x = \sin^2\theta$ and $dx = 2\sin\theta\cos\theta\,d\theta = \sin 2\theta\,d\theta$, we get

$$\int \sin^{-1}\sqrt{x}\,dx = \int \underset{\text{I}}{\theta}\underset{\text{II}}{\sin 2\theta}\,d\theta$$

$$= \theta\left(\frac{-\cos 2\theta}{2}\right) - \int 1\cdot\frac{(-\cos 2\theta)}{2}d\theta$$

[integrating by parts]

$$= -\frac{\theta}{2}\cos 2\theta + \int\frac{1}{2}\cos 2\theta\,d\theta$$

$$= -\frac{1}{2}\theta\cos 2\theta + \frac{1}{4}\sin 2\theta$$

$$= -\frac{1}{2}\theta(1-2\sin^2\theta) + \frac{1}{2}\sin\theta\sqrt{1-\sin^2\theta}$$

$$= -\frac{1}{2}\sin^{-1}\sqrt{x}(1-2x) + \frac{1}{2}\sqrt{x}\cdot\sqrt{1-x}. \quad \ldots \text{(ii)}$$

From (i) and (ii), we get

$$I = \frac{4}{\pi}\cdot\left\{-\frac{1}{2}\sin^{-1}\sqrt{x}(1-2x) + \frac{1}{2}\sqrt{x}\sqrt{1-x}\right\} - x + C$$

$$\therefore \quad I = \frac{-2}{\pi}\sin^{-1}\sqrt{x}(1-2x) + \frac{2}{\pi}\sqrt{x}\sqrt{1-x} - x + C.$$

EXAMPLE 25 *Evaluate* $\int\frac{\sqrt{x^2+1}\{\log(x^2+1) - 2\log x\}}{x^4}dx.$ **[CBSE 2012, '14C]**

SOLUTION We have

$$I = \int\frac{\sqrt{x^2+1}}{x^4}\cdot\log\left(\frac{x^2+1}{x^2}\right)dx$$

$$= \int\sqrt{1+\frac{1}{x^2}}\cdot\frac{1}{x^3}\log\left(1+\frac{1}{x^2}\right)dx \quad \ldots \text{(i)}$$

Putting $\left(1+\frac{1}{x^2}\right)=t$ and $\frac{-2}{x^3}dx=dt$, i.e., $\frac{1}{x^3}dx=-\frac{1}{2}dt$ in (i), we get

$$I=-\frac{1}{2}\int \underset{\text{I}}{(\log t)}\,\underset{\text{II}}{\sqrt{t}}\,dt$$

$$=-\frac{1}{2}\left\{\frac{2}{3}(\log t)t^{3/2}-\frac{2}{3}\int\left(\frac{1}{t}\times t^{3/2}\right)dt\right\}$$

$$=-\frac{1}{3}t^{3/2}(\log t)+\frac{1}{3}\int t^{1/2}dt$$

$$=-\frac{1}{3}t^{3/2}(\log t)+\frac{2}{9}t^{3/2}+C$$

$$=-\frac{1}{3}t^{3/2}\left\{\log t-\frac{2}{3}\right\}+C$$

$$=-\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{3/2}\left\{\log\left(1+\frac{1}{x^2}\right)-\frac{2}{3}\right\}+C.$$

EXAMPLE 26 *Evaluate* $\int\frac{\sqrt{1-\sin x}}{(1+\cos x)}e^{-x/2}dx.$ **[CBSE 2013C]**

SOLUTION Putting $\frac{-x}{2}=t$, we get $x=-2t$ and $dx=-2dt$.

$$\therefore\quad I=\int\frac{\sqrt{1-\sin x}}{(1+\cos x)}e^{-x/2}dx$$

$$=\int\frac{\sqrt{1-\sin(-2t)}}{\{1+\cos(-2t)\}}e^{t}(-2dt)=-2\int\frac{\sqrt{1+\sin 2t}}{(1+\cos 2t)}e^{t}dt$$

$$=-2\int\frac{\sqrt{\cos^2 t+\sin^2 t+2\sin t\cos t}}{2\cos^2 t}e^{t}dt$$

$$=-2\int\frac{(\cos t+\sin t)}{2\cos^2 t}e^{t}dt=-\int(\sec t+\sec t\tan t)\,e^{t}dt$$

$$=-\int e^{t}\{f(t)+f'(t)\}\,dt,\text{ where } f(t)=\sec t$$

$$=-e^{t}f(t)+C=-e^{-x/2}\sec\left(\frac{-x}{2}\right)+C=-e^{-x/2}\sec\frac{x}{2}+C.$$

Integrals of the form $\int e^{x}[f(x)+f'(x)]dx$

THEOREM 1 $\int e^{x}\{f(x)+f'(x)\}\,dx=e^{x}\cdot f(x)+C.$

PROOF $\int e^{x}\{f(x)+f'(x)\}\,dx=\int e^{x}\cdot f(x)\,dx+\int e^{x}\cdot f'(x)\,dx$

$$=f(x)\cdot\int e^{x}dx-\int\{f'(x)\cdot\int e^{x}dx\}\,dx+\int e^{x}f'(x)\,dx+C$$

[evaluating the first integral by parts]

$$= e^x f(x) - \int e^x f'(x)\,dx + \int e^x f'(x)\,dx + C$$

$$= e^x f(x) + C.$$

$$\therefore \quad \int e^x \{f(x) + f'(x)\}\,dx = e^x f(x) + C.$$

EXAMPLE 1 *Evaluate:*

(i) $\int e^x \left(\frac{1}{x} - \frac{1}{x^2}\right) dx$ *(ii)* $\int e^x \left(\frac{1}{x^2} - \frac{2}{x^3}\right) dx$

(iii) $\int e^x \left\{\sin^{-1} x + \frac{1}{\sqrt{1-x^2}}\right\} dx$ *(iv)* $\int e^x (\tan x + \log \sec x)\,dx$

SOLUTION We have

(i) $I = \int e^x \left\{\frac{1}{x} + \left(-\frac{1}{x^2}\right)\right\} dx$

$= \int e^x \{f(x) + f'(x)\}dx$, where $f(x) = \frac{1}{x}$ and $f'(x) = \frac{-1}{x^2}$

$= e^x \cdot f(x) + C = e^x \cdot \frac{1}{x} + C = \frac{e^x}{x} + C.$

(ii) $I = \int e^x \left\{\frac{1}{x^2} + \left(\frac{-2}{x^3}\right)\right\} dx$

$= \int e^x \{f(x) + f'(x)\}dx$, where $f(x) = \frac{1}{x^2}$ and $f'(x) = \frac{-2}{x^3}$

$= e^x \cdot f(x) + C = e^x \cdot \frac{1}{x^2} + C = \frac{e^x}{x^2} + C.$

(iii) $I = \int e^x \left\{\sin^{-1} x + \frac{1}{\sqrt{1-x^2}}\right\} dx$

$= \int e^x \{f(x) + f'(x)\}\,dx$, where $f(x) = \sin^{-1} x$ and $f'(x) = \frac{1}{\sqrt{1-x^2}}$

$= e^x \cdot f(x) + C = e^x \cdot \sin^{-1} x + C = e^x \sin^{-1} x + C.$

(iv) $I = \int e^x (\tan x + \log \sec x)\,dx$

$= \int e^x \{f(x) + f'(x)\}\,dx$, where $f(x) = \log(\sec x)$

and $f'(x) = \frac{1}{\sec x} \cdot \sec x \tan x = \tan x$

$= e^x f(x) + C = e^x \log(\sec x) + C.$

EXAMPLE 2 *Evaluate* $\int \frac{x\,e^x}{(1+x)^2} dx.$

SOLUTION We have

$$I = \int e^x \cdot \left\{\frac{x}{(1+x)^2}\right\} dx = \int e^x \cdot \left\{\frac{(1+x)-1}{(1+x)^2}\right\} dx$$

$$= \int e^x \cdot \left\{ \frac{(1+x)}{(1+x)^2} - \frac{1}{(1+x)^2} \right\} dx = \int e^x \cdot \left\{ \frac{1}{(1+x)} - \frac{1}{(1+x)^2} \right\} dx$$

$$= \int e^x \cdot \{f(x) + f'(x)\}\, dx, \text{ where } f(x) = \frac{1}{1+x} \text{ and } f'(x) = \frac{-1}{(1+x)^2}$$

$$= e^x \cdot f(x) + C = e^x \cdot \frac{1}{(1+x)} + C = \frac{e^x}{(1+x)} + C.$$

EXAMPLE 3 *Evaluate* $\int e^x \left(\frac{1-\sin x}{1-\cos x} \right) dx.$

SOLUTION We have $I = \int e^x \cdot \left(\frac{1-\sin x}{1-\cos x} \right) dx = \int e^x \cdot \left\{ \frac{1}{(1-\cos x)} - \frac{\sin x}{(1-\cos x)} \right\} dx$

$$= \int e^x \cdot \left\{ \frac{1}{2\sin^2(x/2)} - \frac{2\sin(x/2)\cos(x/2)}{2\sin^2(x/2)} \right\} dx$$

$$= \int e^x \cdot \left\{ \frac{1}{2}\operatorname{cosec}^2 \frac{x}{2} - \cot \frac{x}{2} \right\} dx$$

$$= \int e^x \cdot \left\{ -\cot \frac{x}{2} + \frac{1}{2}\operatorname{cosec}^2 \frac{x}{2} \right\} dx$$

$$= \int e^x \cdot \{f(x) + f'(x)\}\, dx,$$

where $f(x) = -\cot \frac{x}{2}$ and $f'(x) = \frac{1}{2}\operatorname{cosec}^2 \frac{x}{2}$

$$= e^x \cdot f(x) + C = e^x \left(-\cot \frac{x}{2} \right) + C = -e^x \cot \frac{x}{2} + C.$$

EXAMPLE 4 *Evaluate* $\int e^x \cdot \left\{ \frac{2+\sin 2x}{1+\cos 2x} \right\} dx.$

SOLUTION We have

$$I = \int e^x \cdot \left\{ \frac{2+\sin 2x}{1+\cos 2x} \right\} dx$$

$$= \int e^x \cdot \left\{ \frac{2}{(1+\cos 2x)} + \frac{\sin 2x}{(1+\cos 2x)} \right\} dx$$

$$= \int e^x \cdot \left\{ \frac{2}{2\cos^2 x} + \frac{2\sin x \cos x}{2\cos^2 x} \right\} dx = \int e^x \cdot \{\sec^2 x + \tan x\}\, dx$$

$$= \int e^x \cdot \{\tan x + \sec^2 x\}\, dx$$

$$= \int e^x \cdot \{f(x) + f'(x)\}\, dx, \text{ where } f(x) = \tan x \text{ and } f'(x) = \sec^2 x$$

$$= e^x \cdot f(x) + C = e^x \tan x + C.$$

EXAMPLE 5 *Evaluate* $\int \frac{(x^2+1)e^x}{(x+1)^2} dx.$ **[CBSE 2005C, '06]**

SOLUTION We have

$$I = \int e^x \cdot \frac{(x^2+1)}{(x+1)^2}\,dx = \int e^x \cdot \left\{\frac{(x+1)^2 - 2x}{(x+1)^2}\right\} dx$$

$$= \int e^x \cdot \left\{1 - \frac{2x}{(x+1)^2}\right\} dx = \int e^x\,dx - 2\int e^x \cdot \frac{x}{(x+1)^2}\,dx$$

$$= e^x - 2\cdot \int e^x \cdot \frac{\{(x+1)-1\}}{(x+1)^2}\,dx = e^x - 2\cdot \int e^x \cdot \left\{\frac{1}{(x+1)} - \frac{1}{(x+1)^2}\right\} dx$$

$$= e^x - 2\cdot \int e^x \cdot \{f(x) + f'(x)\}\,dx,$$

where $f(x) = \dfrac{1}{(x+1)}$ and $f'(x) = \dfrac{-1}{(x+1)^2}$

$$= e^x - 2e^x \cdot f(x) + C = e^x - 2e^x \cdot \frac{1}{(x+1)} + C$$

$$= e^x \cdot \left\{1 - \frac{2}{(x+1)}\right\} + C = e^x\left(\frac{x-1}{x+1}\right) + C.$$

EXAMPLE 6 *Evaluate* $\int e^{2x}\left(\dfrac{\sin 4x - 2}{1 - \cos 4x}\right) dx.$ **[CBSE 2007]**

SOLUTION Putting $2x = t$ and $dx = \dfrac{1}{2}\,dt$, we get

$$I = \frac{1}{2}\int e^t\left(\frac{\sin 2t - 2}{1 - \cos 2t}\right) dt = \frac{1}{2}\int e^t\left(\frac{2\sin t\cos t - 2}{2\sin^2 t}\right) dt$$

$$= \frac{1}{2}\int e^t\left\{\frac{\sin t\cos t - 1}{\sin^2 t}\right\} dt = \frac{1}{2}\int e^t(\cot t - \operatorname{cosec}^2 t)\,dt$$

$$= \frac{1}{2}\int e^t\{f(t) + f'(t)\}\,dt, \quad \text{where } f(t) = \cot t$$

$$= \frac{1}{2}e^t \cdot f(t) + C = \frac{1}{2}e^t \cot t + C = \frac{1}{2}e^{2x}\cot 2x + C.$$

Integrals of the form $\int e^{kx}\cdot\{k\cdot f(x) + f'(x)\}\,dx$

THEOREM 2 $\int e^{kx}\{k\cdot f(x) + f'(x)\}\,dx = e^{kx}\cdot f(x) + C.$

PROOF $\int e^{kx}\cdot\{k\cdot f(x) + f'(x)\}\,dx$

$$= k\cdot\int e^{kx} f(x)\,dx + \int e^{kx} f'(x)\,dx$$

$$= k\cdot\left[f(x)\cdot\frac{e^{kx}}{k} - \int f'(x)\cdot\frac{e^{kx}}{k}\,dx\right] + \int e^{kx} f'(x)\,dx + C$$

[evaluating the first integral by parts]

$$= e^{kx}\cdot f(x) - \int e^{kx}\cdot f'(x)\,dx + \int e^{kx} f'(x)\,dx + C = e^{kx}\cdot f(x) + C.$$

$\therefore \quad \int e^{kx}\cdot\{k\cdot f(x) + f'(x)\}\,dx = e^{kx}\cdot f(x) + C.$

EXAMPLE 7 *Evaluate* $\int e^{2x} \cdot (-\sin x + 2\cos x)\, dx.$

SOLUTION We have

$$I = \int e^{2x} \cdot \{2\cos x - \sin x\}\, dx = 2\int e^{2x} \cos x\, dx - \int e^{2x} \sin x\, dx$$

$$= 2 \cdot \left[\cos x \cdot \frac{e^{2x}}{2} - \int (-\sin x) \cdot \frac{e^{2x}}{2} dx \right] - \int e^{2x} \sin x\, dx$$

[integrating $e^{2x} \cos x$ by parts]

$$= e^{2x} \cos x + \int e^{2x} \sin x\, dx - \int e^{2x} \sin x\, dx + C$$

$$= e^{2x} \cos x + C.$$

Integrals of the form $e^{ax} \cos(bx + c)$ and $e^{ax} \sin(bx + c)$

EXAMPLE 8 *Evaluate* $\int e^{ax} \cos(bx + c)\, dx.$ **[CBSE 2002]**

SOLUTION Integrating by parts, taking e^{ax} as the second function, we get

$$\int e^{ax} \cos(bx + c)\, dx = \cos(bx + c) \cdot \frac{e^{ax}}{a} - \int \left\{ -b \sin(bx + c) \cdot \frac{e^{ax}}{a} \right\} dx$$

$$= \frac{e^{ax}}{a} \cos(bx + c) + \frac{b}{a} \int e^{ax} \sin(bx + c)\, dx$$

$$= \frac{e^{ax}}{a} \cdot \cos(bx + c) + \frac{b}{a} \left[\sin(bx + c) \cdot \frac{e^{ax}}{a} - \int \left\{ b \cos(bx + c) \cdot \frac{e^{ax}}{a} \right\} \right] dx + C$$

[integrating $e^{ax} \sin(bx + c)$ by parts]

$$= \frac{e^{ax}}{a} \cdot \cos(bx + c) + \frac{b}{a^2} e^{ax} \sin(bx + c) - \frac{b^2}{a^2} \int e^{ax} \cos(bx + c)\, dx + C$$

$$\therefore \left(1 + \frac{b^2}{a^2}\right) \int e^{ax} \cos(bx + c)\, dx = \frac{e^{ax}}{a} \cos(bx + c) + \frac{b}{a^2} e^{ax} \sin(bx + c) + C$$

$$\text{or } \int e^{ax} \cos(bx + c)\, dx = e^{ax} \left[\frac{a \cos(bx + c) + b \sin(bx + c)}{(a^2 + b^2)} \right] + C'.$$

REMARK Put $a = r\cos\theta$ and $b = r\sin\theta$ so that

$$r = \sqrt{a^2 + b^2} \text{ and } \theta = \tan^{-1}\left(\frac{b}{a}\right).$$

$$\therefore \int e^{ax} \cos(bx + c)\, dx = \frac{r e^{ax} \cos(bx + c - \theta)}{(a^2 + b^2)}$$

$$= e^{ax} \cdot \frac{\cos[bx + c - \tan^{-1}(b/a)]}{\sqrt{a^2 + b^2}}.$$

Similarly, $\int e^{ax} \sin(bx + c)\, dx = e^{ax} \cdot \frac{\sin[bx + c - \tan^{-1}(b/a)]}{\sqrt{a^2 + b^2}}.$

EXERCISE 13C

Evaluate the following integrals:

1. $\int x e^x dx$
2. $\int x \cos x \, dx$
3. $\int x e^{2x} dx$
4. $\int x \sin 3x \, dx$
5. $\int x \cos 2x \, dx$
6. $\int x \log 2x \, dx$ **[CBSE 2007]**
7. $\int x \operatorname{cosec}^2 x \, dx$
8. $\int x^2 \cos x \, dx$
9. $\int x \sin^2 x \, dx$
10. $\int x \tan^2 x \, dx$
11. $\int x^2 e^x dx$
12. $\int x^2 \cos^3 x \, dx$
13. $\int x^2 e^{3x} dx$
14. $\int x^2 \sin^2 x \, dx$
15. $\int x^3 \log 2x \, dx$
16. $\int x \cdot \log (x+1) \, dx$ **[CBSE 2008C]**
17. $\int \frac{\log x}{x^n} dx$
18. $\int 2x^3 e^{x^2} dx$
19. $\int x \sin^3 x \, dx$
20. $\int x \cos^3 x \, dx$
21. $\int x^3 \cos x^2 dx$
22. $\int \sin x \log (\cos x) \, dx$
23. $\int x \sin x \cos x \, dx$
24. $\int \cos \sqrt{x} \, dx$
25. $\int \operatorname{cosec}^3 x \, dx$
26. $\int x \sin^3 x \cos x \, dx$
27. $\int \sin x \log (\cos x) \, dx$
28. $\int \frac{\log (\log x)}{x} dx$
29. $\int \log (2 + x^2) \, dx$ **[CBSE 2001]**
30. $\int \frac{x}{(1+\sin x)} dx$
31. $\int \left\{ \frac{1}{\log x} - \frac{1}{(\log x)^2} \right\} dx$ **[CBSE 2005]**
32. $\int e^{-x} \cos 2x \cos 4x \, dx$
33. $\int e^{\sqrt{x}} dx$
34. $\int e^{\sin x} \sin 2x \, dx$
35. $\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$ **[CBSE 2012]**
36. $\int \frac{x^2 \tan^{-1} x}{(1+x^2)} dx$
37. $\int \frac{\log (x+2)}{(x+2)^2} dx$
38. $\int x \sin^{-1} x \, dx$ **[CBSE 2000C, '09]**
39. $\int x \cos^{-1} x \, dx$
40. $\int \cot^{-1} x \, dx$
41. $\int x \cot^{-1} x \, dx$
42. $\int x^2 \cot^{-1} x \, dx$ **[CBSE 2006C]**
43. $\int \sin^{-1} \sqrt{x} \, dx$
44. $\int \cos^{-1} \sqrt{x} \, dx$
45. $\int \cos^{-1} (4x^3 - 3x) \, dx$
46. $\int \cos^{-1} \left(\frac{1-x^2}{1+x^2} \right) dx$
47. $\int \tan^{-1} \left(\frac{2x}{1-x^2} \right) dx$

48. $\int \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)dx$

49. $\int \frac{\sin^{-1}x}{x^2}dx$ **[CBSE 2002C, '04]**

50. $\int \frac{\tan x \sec^2 x}{(1-\tan^2 x)}dx$

51. $\int e^{3x} \sin 4x\, dx$

52. $\int e^{2x} \sin x\, dx$

53. $\int e^{2x} \sin x \cos x\, dx$

54. $\int e^{2x} \cos(3x+4)\, dx$

55. $\int e^{-x} \cos x\, dx$

56. $\int e^x (\sin x + \cos x)\, dx$

57. $\int e^x (\cot x - \operatorname{cosec}^2 x)\, dx$

58. $\int e^x \sec x(1+\tan x)\, dx$ **[CBSE 2012]**

59. $\int e^x \left(\tan^{-1}x + \frac{1}{1+x^2}\right)dx$

60. $\int e^x (\cot x + \log \sin x)\, dx$

61. $\int e^x (\tan x - \log \cos x)\, dx$ **[CBSE 2000]**

62. $\int e^x [\sec x + \log(\sec x + \tan x)]dx$

63. $\int e^x \left(\frac{1+\sin x \cos x}{\cos^2 x}\right)dx$

64. $\int e^x \left(\frac{\sin x \cos x - 1}{\sin^2 x}\right)dx$

65. $\int e^x \left(\frac{\cos x + \sin x}{\cos^2 x}\right)dx$

66. $\int e^x \left(\frac{2-\sin 2x}{1-\cos 2x}\right)dx$

67. $\int e^x \left(\frac{1+\sin x}{1+\cos x}\right)dx$

68. $\int e^x \left(\frac{\sin 4x - 4}{1-\cos 4x}\right)dx$ **[CBSE 2010]**

69. $\int \frac{e^x[\sqrt{1-x^2}\sin^{-1}x + 1]}{\sqrt{1-x^2}}dx$

70. $\int e^x \left(\frac{1+x\log x}{x}\right)dx$

71. $\int e^x \cdot \frac{x}{(1+x)^2}dx$

72. $\int e^x \frac{(x-1)}{(x+1)^3}dx$ **[CBSE 2006]**

73. $\int e^x \frac{(2-x)}{(1-x)^2}dx$

74. $\int e^x \cdot \frac{(x-3)}{(x-1)^3}dx$

75. $\int e^{3x}\left(\frac{3x-1}{9x^2}\right)dx$

76. $\int \frac{(x+1)}{(x+2)^2}e^x dx$

77. $\int \frac{xe^{2x}}{(1+2x)^2}dx$

78. $\int e^{2x}\left(\frac{2x-1}{4x^2}\right)dx$

79. $\int e^x \left(\log x + \frac{1}{x^2}\right)dx$

80. $\int \frac{\log x}{(1+\log x)^2}dx$

81. $\int \{\sin(\log x) + \cos(\log x)\}\, dx$

82. $\int \left\{\frac{1}{\log x} - \frac{1}{(\log x)^2}\right\}dx$ **[CBSE 2005]**

83. $\int \left\{\log(\log x) + \frac{1}{(\log x)^2}\right\}dx$

84. $\int \left(\frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}\right)dx$

85. $\int 5^{5^{5^x}} \cdot 5^{5^x} \cdot 5^x\, dx$

86. $\int e^{2x}\left(\frac{1+\sin 2x}{1+\cos 2x}\right)dx$ **[CBSE 2010]** 87. $\int e^{2x}\left(\frac{1-\sin 2x}{1-\cos 2x}\right)dx$ **[CBSE 2013C]**

ANSWERS (EXERCISE 13C)

1. $e^x(x-1)+C$

2. $x\sin x+\cos x+C$

3. $\frac{1}{2}xe^{2x}-\frac{1}{4}e^{2x}+C$

4. $\frac{-x\cos 3x}{3}+\frac{\sin 3x}{9}+C$

5. $\frac{1}{2}x\sin 2x+\frac{1}{4}\cos 2x+C$

6. $\frac{1}{2}x^2\log 2x-\frac{1}{4}x^2+C$

7. $-x\cot x+\log|\sin x|+C$

8. $x^2\sin x+2x\cos x-2\sin x+C$

9. $\frac{x^2}{4}-\frac{1}{4}x\sin 2x-\frac{1}{8}\cos 2x+C$

10. $x\tan x+\log|\cos x|-\frac{x^2}{2}+C$

11. $e^x(x^2-2x+2)+C$

12. $\frac{2x}{9}\cos 3x+\frac{\sin 3x}{36}(4+3x^2)$

13. $e^{3x}\left(\frac{x^2}{3}-\frac{2x}{9}+\frac{2}{27}\right)+C$

14. $\frac{1}{6}x^3-\frac{1}{4}x^2\sin 2x-\frac{1}{4}x\cos 2x+\frac{1}{8}\sin 2x+C$

15. $\frac{x^4\log 2x}{4}-\frac{x^4}{16}+C$

16. $\frac{1}{2}(x^2-1)\log(x+1)-\frac{1}{4}x^2+\frac{1}{2}x+C$

17. $\frac{x^{1-n}}{(1-n)}\cdot\log|x|-\frac{x^{1-n}}{(1-n)^2}+C$

18. $e^{x^2}(x^2-1)+C$

19. $\frac{-3x\cos x}{4}+\frac{3\sin x}{4}+\frac{x\cos 3x}{12}-\frac{\sin 3x}{36}+C$

20. $\frac{1}{12}x\sin 3x+\frac{1}{36}\cos 3x+\frac{3}{4}x\sin x+\frac{3}{4}\cos x+C$

21. $\frac{1}{2}x^2\sin x^2+\frac{1}{2}\cos x^2+C$

22. $-\cos x\log(\cos x)+\cos x+C$

23. $-\frac{1}{4}x\cos 2x+\frac{1}{8}\sin 2x+C$

24. $2[\sqrt{x}\sin\sqrt{x}+\cos\sqrt{x}]+C$

25. $-\frac{1}{2}\operatorname{cosec}x\cot x+\frac{1}{2}\log\left|\tan\frac{x}{2}\right|+C$

26. $\frac{1}{4}x\sin^4x-\frac{3}{32}x+\frac{1}{16}\sin 2x-\frac{1}{128}\sin 4x+C$

27. $\cos x(1-\log\cos x)+C$

28. $(\log x)[\log(\log x)-1]+C$

29. $x\log(x^2+2)-2x+2\sqrt{2}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+C$

30. $-x\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)+2\log\left|\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)\right|+C$

31. $\frac{x}{(\log x)}+C$

32. $e^{-x}\left[\frac{1}{74}(6\sin 6x-\cos 6x)+\frac{1}{10}(2\sin 2x-\cos 2x)\right]+C$

33. $2e^{\sqrt{x}}(\sqrt{x}-1)+C$ **34.** $2e^{\sin x}(\sin x-1)+C$ **35.** $-\sqrt{1-x^2}\sin^{-1}x+x+C$

36. $x\tan^{-1}x-\frac{1}{2}\log(1+x^2)-\frac{1}{2}(\tan^{-1}x)^2+C$ **37.** $\frac{-1}{(x+2)}-\frac{\log(x+2)}{(x+2)}+C$

38. $\frac{1}{2}x^2\sin^{-1}x+\frac{1}{4}x\sqrt{1-x^2}-\frac{1}{4}\sin^{-1}x+C$

39. $\frac{1}{2}x^2\cos^{-1}x-\frac{1}{4}x\sqrt{1-x^2}+\frac{1}{4}\sin^{-1}x+C$ **40.** $x\cot^{-1}x+\frac{1}{2}\log(1+x^2)+C$

41. $\frac{1}{2}x^2(\cot^{-1}x)+\frac{1}{2}(x-\tan^{-1}x)+C$

42. $\frac{1}{3}x^3\cot^{-1}x+\frac{1}{6}x^2-\frac{1}{6}\log|x^2+1|+C$

43. $\frac{1}{2}(2x-1)\sin^{-1}\sqrt{x}+\frac{1}{2}\sqrt{x(1-x)}+C$

44. $\frac{1}{2}(2x-1)\cos^{-1}\sqrt{x}-\frac{1}{2}\sqrt{x(1-x)}+C$ **45.** $3x\cos^{-1}x-3\sqrt{1-x^2}+C$

46. $2x\tan^{-1}x-\log(1+x^2)+C$ **47.** $2x\tan^{-1}x-\log(1+x^2)+C$

48. $3x\tan^{-1}x-\frac{3}{2}\log(1+x^2)+C$ **49.** $\frac{-\sin^{-1}x}{x}+\log\left|\frac{1}{x}-\frac{\sqrt{1-x^2}}{x}\right|+C$

50. $-\frac{1}{2}\log|1-\tan^2x|+C$ **51.** $\frac{e^{3x}}{25}\cdot(3\sin 4x-4\cos 4x)+C$

52. $\frac{1}{5}\cdot e^{2x}(2\sin x-\cos x)+C$ **53.** $\frac{1}{8}\cdot e^{2x}(\sin 2x-\cos 2x)+C$

54. $\frac{e^{2x}}{13}\cdot\{2\cos(3x+4)+3\sin(3x+4)\}+C$

55. $\frac{1}{2}e^{-x}(\sin x-\cos x)+C$ **56.** $e^x\sin x+C$

57. $e^x\cot x+C$ **58.** $e^x\sec x+C$ **59.** $e^x\tan^{-1}x+C$

60. $e^x\log(\sin x)+C$ **61.** $e^x\log(\sec x)+C$

62. $e^x\log(\sec x+\tan x)+C$ **63.** $e^x\tan x+C$ **64.** $e^x\cot x+C$

65. $e^x\sec x+C$ **66.** $-e^x\cot x+C$ **67.** $e^x\tan\frac{x}{2}+C$ **68.** $e^x\cot 2x+C$

69. $e^x\sin^{-1}x+C$ **70.** $e^x\log x+C$ **71.** $\frac{e^x}{(1+x)}+C$ **72.** $\frac{e^x}{(x+1)^2}+C$

73. $\frac{e^x}{(1-x)}+C$ **74.** $\frac{e^x}{(x-1)^2}+C$ **75.** $\frac{e^{3x}}{9x}+C$ **76.** $\frac{e^x}{(x+2)}+C$

77. $\frac{e^{2x}}{4(1+2x)}+C$ **78.** $\frac{e^{2x}}{4x}+C$ **79.** $e^x\left(\log x-\frac{1}{x}\right)+C$

80. $\dfrac{x}{(1+\log x)}+C$ **81.** $e^{\log x}\sin(\log x)+C$

82. $\dfrac{x}{(\log x)}+C$ **83.** $x\cdot\left\{\log(\log x)-\dfrac{1}{\log x}\right\}+C$

84. $\dfrac{2}{\pi}\{\sqrt{x-x^2}-(1-2x)\sin^{-1}\sqrt{x}\}-x+C$ **85.** $\dfrac{5^{5^{5^x}}}{(\log 5)^3}+C$

86. $\dfrac{1}{2}e^{2x}\tan x+C$ **87.** $\dfrac{1}{2}e^{2x}\tan x+C$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 13C)

9. Write $\sin^2 x=\dfrac{1}{2}(1-\cos 2x)$. 10. $\tan^2 x=(\sec^2 x-1)$.

15. Integrate by parts, taking $(\log 2x)$ as the 2nd function.

16. Integrate by parts, taking x as the second function.

18. Put $x^2=t$. 19. $\sin^3 x=\dfrac{3}{4}\sin x-\dfrac{1}{4}\sin 3x$.

20. $\cos^3 x=\dfrac{1}{4}\cos 3x+\dfrac{3}{4}\cos x$. 21. Put $x^2=t$. 22. Put $\cos x=t$.

23. $I=\dfrac{1}{2}\int x\sin 2x\,dx$. 24. Put $\sqrt{x}=t$ and $dx=2t\,dt$.

25. $I=\int \operatorname{cosec} x(\operatorname{cosec}^2 x)\,dx$. Now, integrate by parts.

26. Take $(\sin^3 x\cos x)$ as 2nd function. Its integral is $\dfrac{1}{4}\sin^4 x$.

Further, use $\sin^4 x=\left(\dfrac{1-\cos 2x}{2}\right)^2$.

27. Put $\cos x=t$. 29. Take $\log(2+x^2)$ as 1st function.

30. $I=\displaystyle\int\frac{x}{1+\cos\left(\frac{\pi}{2}-x\right)}dx=\frac{1}{2}\int x\sec^2\left(\frac{\pi}{4}-\frac{x}{2}\right)dx$

31. Integrate $\dfrac{1}{(\log x)}$ by parts, taking 1 as 2nd function.

32. $\cos 2x\cos 4x=\dfrac{1}{2}(\cos 6x+\cos 2x)$. 33. Put $\sqrt{x}=t$ and $dx=2t\,dt$.

34. $I=2\int e^{\sin x}\sin x\cos x\,dx=2\int t\,e^t dt$, where $\sin x=t$.

35. Put $x=\sin t$ and $dx=\cos t\,dt$.

36. Put $x=\tan t$, $dx=\sec^2 t\,dt$ and then write $\tan^2 t=(\sec^2 t-1)$.

37. Integrate by parts, taking $\log(x+2)$ as the 1st function and $\dfrac{1}{(x+2)^2}$ as the 2nd function.

38. Integrate by parts, taking $\sin^{-1}x$ as the 1st function and x as 2nd.

$$I = (\sin^{-1}x)\cdot\frac{x^2}{2} - \int \frac{1}{1-x^2}\cdot\frac{x^2}{2}\,dx = \frac{1}{2}x^2(\sin^{-1}x) + \frac{1}{2}\cdot\int\frac{(1-x^2)-1}{\sqrt{1-x^2}}\,dx$$

$$= \frac{1}{2}x^2\,(\sin^{-1}x) + \frac{1}{2}\cdot\int\sqrt{1-x^2}\,dx - \frac{1}{2}\int\frac{dx}{\sqrt{1-x^2}}$$

$$= \frac{1}{2}x^2(\sin^{-1}x) + \frac{1}{2}\cdot\left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1}x\right] - \frac{1}{2}\sin^{-1}x + C.$$

43. Put $\sqrt{x} = \sin t$. Then, $x = \sin^2 t$ and $dx = \sin 2t\ dt$.

$$I = \int t\sin 2t\ dt = -\frac{1}{2}t\cos 2t + \frac{1}{4}\sin 2t + C$$

$$= \frac{1}{2}t\,(2\sin^2 t - 1) + \frac{1}{2}\sin t\cos t = \frac{1}{2}\sin^{-1}\sqrt{x}\,(2x-1) + \frac{1}{2}\sqrt{x(1-x)} + C.$$

45. Put $x = \cos t$ and $dx = -\sin t\ dt$. Use $(4\cos^3 t - 3\cos t) = \cos 3t$.

46. Put $x = \tan t$ and $dx = \sec^2 t\ dt$. Use $\left(\frac{1-\tan^2 t}{1+\tan^2 t}\right) = \cos 2t$

47. Put $x = \tan t$ and $dx = \sec^2 t\ dt$.

48. Put $x = \tan t$ and $dx = \sec^2 t\ dt$. Use $\left(\frac{3\tan t - \tan^3 t}{1-3\tan^2 t}\right) = \tan 3t$.

49. Put $x = \sin t$ and $dx = \cos t\ dt$. Then,

$$I = \int t(\operatorname{cosec} t\cot t)\,dt = -t\operatorname{cosec} t + \log|\operatorname{cosec} t - \cot t| + C$$

$$= \frac{-\sin^{-1}x}{x} + \log\left|\frac{1}{x} - \frac{\sqrt{1-x^2}}{x}\right| + C.$$

50. Put $\tan x = t$ and $\sec^2 x\,dx = dt$.

51. Integrating by parts, taking e^{3x} as 2nd function, we get

$$I = (\sin 4x)\cdot\frac{e^{3x}}{3} - \int 4\cos 4x\cdot\frac{e^{3x}}{3}\,dx = \frac{1}{3}e^{3x}(\sin 4x) - \frac{4}{3}\cdot\int e^{3x}\cos 4x\ dx$$

$$= \frac{1}{3}e^{3x}(\sin 4x) - \frac{4}{3}\cdot\left[(\cos 4x)\cdot\frac{e^{3x}}{3} - \int(-4\sin 4x) - \frac{e^{3x}}{3}\,dx\right]$$

$$= \frac{1}{3}e^{3x}(\sin 4x) - \frac{4}{9}e^{3x}(\cos 4x) - \frac{16}{9}I$$

$$\Leftrightarrow \left(1 + \frac{16}{9}\right)I = \frac{1}{3}e^{3x}(\sin 4x) - \frac{4}{9}e^{3x}(\cos 4x).$$

53. $I = \frac{1}{2}\cdot\int e^{2x}\sin 2x\,dx.$

61. $I = \int e^x(\log\sec x + \tan x)\,dx = \int e^x\{f(x) + f'(x)\}\,dx$, where $f(x) = \log\sec x$.

62. $I = \int e^x[\log(\sec x + \tan x) + \sec x]\,dx = \int e^x\{f(x) + f'(x)\}\,dx$,
where $f(x) = \log(\sec x + \tan x)$.

63. $I = \int e^x(\sec^2 x + \tan x)\,dx = \int e^x(\tan x + \sec^2 x)\,dx = e^x\tan x + C.$

64. $I = \int e^x(\cot x - \operatorname{cosec}^2 x)\,dx = e^x\cot x + C.$

65. $I = \int e^x(\sec x + \sec x \tan x)\,dx = e^x \sec x + C.$

66. $I = \int e^x\left\{\frac{2}{2\sin^2 x} - \frac{2\sin x \cos x}{2\sin^2 x}\right\}dx = \int e^x(-\cot x + \operatorname{cosec}^2 x)\,dx = -e^x \cot x + C.$

67. $I = \int e^x\left\{\frac{1}{2\cos^2(x/2)} + \frac{2\sin(x/2)\cos(x/2)}{2\cos^2(x/2)}\right\}dx$

$= \int e^x \cdot \left\{\frac{1}{2}\sec^2\frac{x}{2} + \tan\frac{x}{2}\right\}dx = \int e^x\left\{\tan\frac{x}{2} + \frac{1}{2}\sec^2\frac{x}{2}\right\}dx = e^x \tan\frac{x}{2} + C.$

68. $I = \int e^x\left\{\frac{2\sin 2x\cos 2x}{2\sin^2 2x} - \frac{4}{2\sin^2 2x}\right\}dx = \int e^x\{\cot 2x - 2\operatorname{cosec}^2 2x\}\,dx = e^x \cot 2x + C.$

69 $I = \int e^x\left\{\sin^{-1}x + \frac{1}{\sqrt{1-x^2}}\right\}dx = e^x \sin^{-1}x + C.$

70. $I = \int e^x\left(\frac{1}{x} + \log x\right) = \int e^x\left(\log x + \frac{1}{x}\right)dx = e^x \log x + C.$

71. $I = \int e^x \frac{(1+x-1)}{(1+x)^2}dx = \int e^x\left[\frac{1}{(1+x)} - \frac{1}{(1+x)^2}\right]dx$

$= \int e^x\{f(x) + f'(x)\}\,dx$, where $f(x) = \frac{1}{(1+x)}$.

72. $I = \int \frac{e^x(x+1-2)}{(x+1)^3}dx = \int e^x\left\{\frac{(x+1)}{(x+1)^3} - \frac{2}{(x+1)^3}\right\}dx$

$= \int e^x\left\{\frac{1}{(x+1)^2} - \frac{2}{(x+1)^3}\right\}dx = \int e^x \cdot \{f(x) + f'(x)\}\,dx$, where $f(x) = \frac{1}{(x+1)^2}$.

73. $I = \int e^x \frac{(1-x+1)}{(1-x)^2}dx = \int e^x\left\{\frac{(1-x)}{(1-x)^2} + \frac{1}{(1-x)^2}\right\}dx$

$= \int e^x\left\{\frac{1}{(1-x)} + \frac{1}{(1-x)^2}\right\}dx = \int e^x \cdot \{f(x) + f'(x)\}\,dx$, where $f(x) = \frac{1}{(1-x)}$.

74. $I = \int \frac{e^x(x-1-2)}{(x-1)^3}dx = \int e^x\left\{\frac{(x-1)}{(x-1)^3} - \frac{2}{(x-1)^3}\right\}dx$

$= \int e^x\left\{\frac{1}{(x-1)^2} - \frac{2}{(x-1)^3}\right\}dx = \int e^x \cdot \{f(x) + f'(x)\}\,dx$, where $f(x) = \frac{1}{(x-1)^2}$.

75. Putting $3x = t$ and $dx = \frac{1}{3}dt$, we get

$$I = \frac{1}{3} \cdot \int e^t\left(\frac{t-1}{t^2}\right)dt = \frac{1}{3} \cdot \int e^t\left(\frac{1}{t} - \frac{1}{t^2}\right)dt = \frac{1}{3}e^t \cdot \frac{1}{t} + C = \frac{e^{3x}}{9x} + C.$$

76. $I = \int e^x \cdot \left\{\frac{(x+2)-1}{(x+2)^2}\right\}dx = \int e^x \cdot \left\{\frac{1}{(x+2)} - \frac{1}{(x+2)^2}\right\}dx = e^x \cdot \frac{1}{(x+2)} + C.$

77. Putting $2x = t$ and $dx = \frac{1}{2}dt$, we get

$$I = \frac{1}{2}\cdot\int \frac{2x\cdot e^{2x}}{(1+2x)^2}dx = \frac{1}{4}\cdot\int \frac{te^t}{(1+t)^2}dt = \frac{1}{4}\cdot\int \frac{(1+t-1)}{(1+t)^2}e^t dt$$

$$= \frac{1}{4}\cdot e^t\cdot\left\{\int \frac{1}{1+t} - \frac{1}{(1+t)^2}\right\}dt = \frac{e^t}{4(1+t)} + C = \frac{e^{2x}}{4(1+2x)} + C.$$

78. Putting $2x = t$ and $dx = \frac{1}{2}dt$, we get

$$I = \frac{1}{2}\cdot\int e^t\cdot\left(\frac{t-1}{t^2}\right)dt = \frac{1}{2}\cdot\int e^t\cdot\left\{\frac{1}{t} - \frac{1}{t^2}\right\}dt = \frac{1}{2}e^t\cdot\frac{1}{t} + C = \frac{1}{4}\cdot\frac{e^{2x}}{x} + C.$$

79. $I = \int e^x\left\{(\log x) + \frac{1}{x} - \frac{1}{x} + \frac{1}{x^2}\right\}dx = \int e^x\left\{\log x + \frac{1}{x}\right\}dx - \int e^x\left(\frac{1}{x} - \frac{1}{x^2}\right)dx$

$$= e^x(\log x) - e^x\cdot\frac{1}{x} + C = e^x\left(\log x - \frac{1}{x}\right) + C.$$

80. Putting $\log x = t$, $x = e^t$ and $dx = e^t dt$, we get

$$I = \int \frac{t}{(1+t)^2}\cdot e^t dt = \int e^t\cdot\frac{(1+t-1)}{(1+t)^2}dt = \int e^t\cdot\left\{\frac{1}{(1+t)} - \frac{1}{(1+t)^2}\right\}dt$$

$$= e^t\frac{1}{(1+t)} + C = \frac{x}{(1+\log x)} + C.$$

81. Putting $\log x = t$, $x = e^t$ and $dx = e^t dt$, we get

$$I = \int e^t(\sin t + \cos t)\,dt = e^t\sin t + C = e^{\log x}\cdot\sin(\log x) + C.$$

82. Putting $\log x = t$, $x = e^t$ and $dx = e^t dt$, we get

$$I = \int e^t\left(\frac{1}{t} - \frac{1}{t^2}\right)dt = e^t\cdot\frac{1}{t} + C = \frac{x}{\log x} + C.$$

83. Putting $\log x = t$, $x = e^t$ and $dx = e^t dt$, we get

$$I = \int\left\{\log t + \frac{1}{t^2}\right\}e^t dt = \int e^t\left\{\log t + \frac{1}{t} - \frac{1}{t} + \frac{1}{t^2}\right\}dt$$

$$= \int e^t\left(\log t + \frac{1}{t}\right)dt - \int e^t\left(\frac{1}{t} - \frac{1}{t^2}\right)dt = e^t\log t - e^t\cdot\frac{1}{t} + C$$

$$= x\left\{\log(\log x) - \frac{1}{\log x}\right\} + C.$$

84. $I = \int \frac{\sin^{-1}\sqrt{x} - \left(\frac{\pi}{2} - \sin^{-1}x\right)}{(\pi/2)}dx = \frac{2}{\pi}\cdot\int\left(2\sin^{-1}\sqrt{x} - \frac{\pi}{2}\right)dx$

$$= \frac{4}{\pi}\int \sin^{-1}\sqrt{x}\,dx - \int dx = \frac{4}{\pi}\cdot\int \sin^{-1}\sqrt{x}\,dx - x + C$$

$$= \frac{4}{\pi}\cdot[\int t\sin 2t\,dt] - x + C, \text{ where } \sin^{-1}\sqrt{x} = t.$$

85. Putting $5^x = t \Leftrightarrow 5^x\log 5\,dx = dt \Leftrightarrow 5^x dx = \frac{1}{\log 5}dt.$

$$\therefore\ I = \int 5^{5^t} 5^t \frac{1}{\log 5}\,dt = \int 5^u \cdot \frac{1}{(\log 5)^2}\,du, \text{ where } 5^t = u \Leftrightarrow 5^t \log 5\,dt = du$$

$$= \int \frac{1}{(\log 5)^3}\,dv, \text{ where } 5^u = v \Leftrightarrow 5^u \log 5\,du = dv$$

$$= \frac{v}{(\log 5)^3} + C = \frac{5^u}{(\log 5)^3} + C = \frac{5^{5^t}}{(\log 5)^3} + C = \frac{5^{5^{5^x}}}{(\log 5)^3} + C.$$

87. $I = \int e^{2x}\left\{\frac{1 + 2\sin x \cos x}{2\cos^2 x}\right\}dx$

$$= \int e^{2x}\left\{\frac{1}{2\cos^2 x} + \frac{2\sin x \cos x}{2\cos^2 x}\right\}dx$$

$$= \frac{1}{2}\int e^{2x}(\sec^2 x)\,dx + \int e^{2x}\tan x\,dx$$

$$= \frac{1}{2}\{e^{2x}\tan x - \int 2e^{2x}\tan x\,dx\} + \int e^{2x}\tan x\,dx + C$$

$$= \frac{1}{2}e^{2x}\tan x + C.$$

OBJECTIVE QUESTIONS II

Mark (✓) against the correct answer in each of the following:

1. $\int x e^x\,dx = ?$

   (a) $e^x(1-x) + C$ (b) $e^x(x+1) + C$ (c) $e^x(x-1) + C$ (d) none of these

2. $\int x e^{2x}\,dx = ?$

   (a) $\frac{1}{2}x e^{2x} + \frac{1}{4}e^{2x} + C$ (b) $\frac{1}{2}x e^{2x} - \frac{1}{4}e^{2x} + C$

   (c) $2x e^{2x} + 4e^{2x} + C$ (d) none of these

3. $\int x\cos 2x\,dx = ?$

   (a) $\frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + C$ (b) $\frac{1}{2}x\sin 2x - \frac{1}{4}\cos 2x + C$

   (c) $2x\sin 2x + 4\cos 2x + C$ (d) none of these

4. $\int x\sec^2 x\,dx = ?$

   (a) $x\tan x - \log|\cos x| + C$ (b) $x\tan x + \log|\cos x| + C$

   (c) $x\tan x + \log|\sec x| + C$ (d) none of these

5. $\int x\sin 2x\,dx = ?$

   (a) $\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + C$ (b) $-\frac{1}{2}x\cos 2x - \frac{1}{4}\sin 2x + C$

   (c) $-\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + C$ (d) none of these

6. $\int x \log\ x\, dx = ?$

(a) $x \log\ x + \frac{1}{2}x^2 + C$ (b) $\frac{1}{2}x^2 \log\ x + \frac{1}{4}x^2 + C$

(c) $\frac{1}{2}x^2 \log\ x - \frac{1}{4}x^2 + C$ (d) none of these

7. $\int x \operatorname{cosec}^2 x\, dx = ?$

(a) $x \cot\ x - \log\ |\sin\ x| + C$ (b) $-x \cot\ x + \log\ |\sin\ x| + C$

(c) $x \tan\ x - \log\ |\sec\ x| + C$ (d) none of these

8. $\int x \sin\ x\ \cos x\, dx = ?$

(a) $-\frac{1}{4}x \sin\ 2x + \frac{1}{8}\cos\ 2x + C$ (b) $\frac{1}{4}x \cos\ 2x - \frac{1}{8}\sin\ 2x + C$

(c) $\frac{1}{2}x \sin\ 2x + \frac{1}{4}\cos\ 2x + C$ (d) $-\frac{1}{4}x \cos\ 2x + \frac{1}{8}\sin\ 2x + C$

9. $\int x \cos^2 x\, dx = ?$

(a) $\frac{x^2}{4} - \frac{x \sin\ 2x}{4} + \frac{\cos\ 2x}{8} + C$ (b) $\frac{x^2}{4} + \frac{x \sin\ 2x}{4} + \frac{\cos\ 2x}{8} + C$

(c) $\frac{x^2}{4} + \frac{x \sin\ 2x}{4} - \frac{\cos\ 2x}{8} + C$ (d) none of these

10. $\int \frac{\log\ x}{x^2} dx = ?$

(a) $-\frac{1}{x}(\log\ x + 1) + C$ (b) $\frac{1}{x}(\log\ x - 1) + C$

(c) $\frac{1}{x}(\log\ x + 1) + C$ (d) none of these

11. $\int \log\ x\, dx = ?$

(a) $\frac{1}{x} + C$ (b) $\frac{1}{2}(\log\ x)^2 + C$

(c) $x(\log\ x + 1) + C$ (d) $x(\log\ x - 1) + C$

12. $\int \log_{10} x\, dx = ?$

(a) $\frac{1}{x}\log_e 10 + C$ (b) $\frac{1}{x}\log_{10} e + C$

(c) $x(\log\ x - 1)\log_e 10 + C$ (d) $x(\log\ x - 1)\log_{10} e + C$

13. $\int (\log\ x)^2 dx = ?$

(a) $\frac{2 \log\ x}{x} + C$ (b) $\frac{1}{3}(\log\ x)^3 + C$

(c) $x(\log\ x)^2 - 2x \log\ x + 2x + C$ (d) $x(\log\ x)^2 + 2x \log\ x - 2x + C$

14. $\int e^{\sqrt{x}}\,dx = ?$

(a) $e^{\sqrt{x}} + \sqrt{x} + C$ (b) $\frac{1}{2}e^{\sqrt{x}}(\sqrt{x}+1) + C$

(c) $2e^{\sqrt{x}}(\sqrt{x}-1) + C$ (d) none of these

15. $\int \cos\sqrt{x}\,dx = ?$

(a) $\sin\sqrt{x} + \cos\sqrt{x} + C$ (b) $\frac{1}{2}(\sqrt{x}\sin\sqrt{x} - \cos\sqrt{x}) + C$

(c) $2[\sqrt{x}\sin\sqrt{x} + \cos\sqrt{x}] + C$ (d) none of these

16. $\int \cos(\log x)\,dx = ?$

(a) $\frac{x}{2}[\cos(\log x) - \sin(\log x)] + C$ (b) $\frac{x}{2}[\cos(\log x) + \sin(\log x)] + C$

(c) $2x[\cos(\log x) + \sin(\log x)] + C$ (d) $2x[\cos(\log x) - \sin(\log x)] + C$

17. $\int \sec^3 x\,dx = ?$

(a) $\frac{1}{2}\{\sec x\tan x - \log|\sec x + \tan x|\} + C$

(b) $\frac{1}{2}\{\sec x\tan x + \log|\sec x + \tan x|\} + C$

(c) $2\{\sec x\tan x + \log|\sec x + \tan x|\} + C$

(d) none of these

18. $\int\left\{\frac{1}{(\log x)} - \frac{1}{(\log x)^2}\right\}dx = ?$

(a) $x\log x + C$ (b) $\frac{x}{\log x} + C$

(c) $x + \frac{1}{\log x} + C$ (d) none of these

19. $\int 2x^3 e^{x^2}\,dx = ?$

(a) $e^{x^2}(x^2-1) + C$ (b) $e^{x^2}(x^2+1) + C$ (c) $e^{x^2}(x+1) + C$ (d) none of these

20. $\int (x2^x)\,dx = ?$

(a) $\frac{2^x}{(\log 2)}(x + \log 2) + C$ (b) $\frac{2^x}{(\log 2)^2}(x\log 2 - 1) + C$

(c) $\frac{x\cdot 2^x}{(\log 2)} + \frac{2^x}{(\log 2)^2} + C$ (d) none of these

21. $\int x\cot^2 x\,dx = ?$

(a) $-x\cot x + \frac{x^2}{2} + \log|\sin x| + C$ (b) $-x\cot x - \frac{x^2}{2} + \log|\sin x| + C$

(c) $-x\cot x + \frac{x^2}{2} - \log|\sin x| + C$ (d) none of these

22. $\int \sin\sqrt{x}\,dx = ?$

(a) $-\sqrt{x}\cos\sqrt{x} + C$ (b) $-\sqrt{x}\cos\sqrt{x} - 2\sin\sqrt{x} - C$

(c) $-2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C$ (d) none of these

23. $\int e^{\sin x}\sin 2x\,dx = ?$

(a) $(2\sin x)e^{\sin x} + C$ (b) $(2\cos x)e^{\sin x} + C$

(c) $2e^{\sin x}(\sin x + 1) + C$ (d) $2e^{\sin x}(\sin x - 1) + C$

24. $\int \frac{\sin^{-1}x}{(1-x^2)^{3/2}}dx = ?$

(a) $\frac{\sin^{-1}x}{\sqrt{1-x^2}} - \frac{1}{2}\log|1-x^2| + C$ (b) $x\sin^{-1}x + \frac{1}{2}\log|1-x^2| + C$

(c) $\frac{x\sin^{-1}x}{\sqrt{1-x^2}} + \frac{1}{2}\log|1-x^2| + C$ (d) none of these

25. $\int \frac{x\tan^{-1}x}{(1-x^2)^{3/2}}dx = ?$

(a) $\frac{\tan^{-1}x}{\sqrt{1+x^2}} - \frac{x}{\sqrt{1+x^2}} + C$ (b) $\frac{-\tan^{-1}x}{\sqrt{1+x^2}} + \frac{x}{\sqrt{1+x^2}} + C$

(c) $\frac{x\tan^{-1}x}{\sqrt{1+x^2}} + \frac{1}{2}\log\left|\frac{x}{\sqrt{1+x^2}}\right| + C$ (d) none of these

26. $\int x\tan^{-1}x\,dx = ?$

(a) $\frac{1}{2}\tan^{-1}x + \log(1+x^2) - \frac{1}{2}x + C$ (b) $\frac{1}{2}x^2\tan^{-1}x + \frac{1}{2}x + C$

(c) $\frac{1}{2}(1+x^2)\tan^{-1}x - \frac{1}{2}x + C$ (d) none of these

27. $\int \tan^{-1}\sqrt{x}\,dx = ?$

(a) $(x-1)\tan^{-1}\sqrt{x} + \sqrt{x} + C$ (b) $(x+1)\tan^{-1}\sqrt{x} - \sqrt{x} + C$

(c) $\frac{1}{2}\sqrt{x}\tan^{-1}\sqrt{x} - \frac{1}{2}\sqrt{x} + C$ (d) none of these

28. $\int \cos^{-1}x\,dx = ?$

(a) $x\cos^{-1}x - \sqrt{1-x^2} + C$ (b) $x\cos^{-1}x + \sqrt{1-x^2} + C$

(c) $x\sin^{-1}x - \sqrt{1-x^2} + C$ (d) none of these

29. $\int \tan^{-1} x\, dx = ?$

(a) $x \tan^{-1} x + \frac{1}{2}\log |1 + x^2| + C$ (b) $x \tan^{-1} x - \frac{1}{2}\log |1 + x^2| + C$

(c) $-x \tan^{-1} x + \frac{1}{2}\log |1 + x^2| + C$ (d) none of these

30. $\int \sec^{-1} x\, dx = ?$

(a) $x \sec^{-1} x + \log |x + \sqrt{x^2 - 1}| + C$ (b) $x \sec^{-1} x - \log |x + \sqrt{x^2 - 1}| + C$

(c) $x \sec^{-1} x + \log |x - \sqrt{x^2 - 1}| + C$ (d) none of these

31. $\int \sin^{-1}(3x - 4x^3)\, dx = ?$

(a) $3[x \sin^{-1} x + \sqrt{1 - x^2}] + C$ (b) $3[x \sin^{-1} x - \sqrt{1 - x^2}] + C$

(c) $\frac{3x^2}{2} + C$ (d) none of these

32. $\int \sin^{-1}\left(\frac{2x}{1 + x^2}\right) dx = ?$

(a) $2x \tan^{-1} x + \log |1 + x^2| + C$ (b) $2x \tan^{-1} x - \log |1 + x^2| + C$

(c) $2x \sin^{-1} x + \log |1 + x^2| + C$ (d) none of these

33. $\int \tan^{-1} \sqrt{\frac{1 - x}{1 + x}} dx = ?$

(a) $\frac{1}{2} x(\cos^{-1} x) + \frac{1}{2}\sqrt{1 - x^2} + C$ (b) $\frac{1}{2} x(\sin^{-1} x) + \frac{1}{2}\sqrt{1 - x^2} + C$

(c) $\frac{1}{2} x(\cos^{-1} x) - \frac{1}{2}\sqrt{1 - x^2} + C$ (d) none of these

34. $\int \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right) dx = ?$

(a) $3x \tan^{-1} x + \frac{3}{2}\log (1 + x^2) + C$ (b) $3x \tan^{-1} x - \frac{3}{2}\log (1 + x^2) + C$

(c) $3x \cos^{-1} x - \frac{3}{2}\sqrt{1 - x^2} + C$ (d) $3x \sin^{-1} x + \frac{3}{2}\sqrt{1 - x^2} + C$

35. $\int x^2 \cos x\, dx = ?$

(a) $x^2 \sin x + 2x \cos x - 2 \sin x + C$ (b) $2x \cos x - x \sin x + 2 \sin x + C$

(c) $x^2 \sin x - 2x \sin x + 2 \sin x + C$ (d) none of these

36. $\int \sin x \log (\cos x)\, dx = ?$

(a) $\cos x \log (\cos x) - \cos x + C$ (b) $-\cos x \log (\cos x) + \cos x + C$

(c) $\cos x \log (\cos x) + \cos x + C$ (d) none of these

37. $\int x \sin x \cos x \, dx = ?$

(a) $-\frac{1}{4}x\cos 2x + \frac{1}{8}\sin 2x + C$ (b) $\frac{1}{4}x\cos 2x + \frac{1}{8}\sin 2x + C$

(c) $\frac{1}{4}x\cos 2x - \frac{1}{8}\sin 2x + C$ (d) none of these

38. $\int x^3 \cos x^2 dx = ?$

(a) $x^2 \sin x^2 + \cos x^2 + C$ (b) $\frac{1}{2}x^2 \sin x^2 + \frac{1}{2}\cos x^2 + C$

(c) $-\frac{1}{2}x^2 \sin x^2 + \frac{1}{2}\cos x^2 + C$ (d) none of these

39. $\int \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) dx = ?$

(a) $2x\tan^{-1}x + \log(1+x^2) + C$ (b) $-2x\tan^{-1}x - 2\log(1+x^2) + C$

(c) $2x\tan^{-1}x - \log(1+x^2) + C$ (d) none of these

40. $\int x\tan^{-1}x \, dx = ?$

(a) $\frac{1}{2}(x^2+1)\tan^{-1}x - \frac{1}{2}x + C$ (b) $\frac{1}{2}(x^2-1)\tan^{-1}x - \frac{1}{2}x + C$

(c) $\frac{1}{2}(x^2+1)\tan^{-1}x + \frac{1}{2}x + C$ (d) none of these

41. $\int \sin(\log x) dx = ?$

(a) $\frac{1}{2}x\sin\log x + \frac{1}{2}x\cos(\log x) + C$

(b) $\frac{1}{2}x\sin\log x - \frac{1}{2}x\cos(\log x) + C$

(c) $-\frac{1}{2}x\sin(\log x) + \frac{1}{2}x\cos(\log x) + C$

(d) none of these

42. $\int (\sin^{-1}x)^2 dx = ?$

(a) $\frac{2\sin^{-1}x}{\sqrt{1-x^2}} + C$

(b) $\frac{1}{3}(\sin^{-1}x)^3 + \frac{1}{\sqrt{1-x^2}} + C$

(c) $x(\sin^{-1}x)^2 + (\sin^{-1}x)\sqrt{1-x^2} + 2x + C$

(d) $x(\sin^{-1}x)^2 + 2(\sin^{-1}x)\sqrt{1-x^2} - 2x + C$

43. $\int e^x \left\{\frac{1}{x} - \frac{1}{x^2}\right\} dx = ?$

(a) $e^x \left\{\log x + \frac{1}{x}\right\} + C$ (b) $xe^x - e^x + C$

(c) $e^x \cdot \frac{1}{x} + C$ (d) none of these

44. $\int e^x \left(\frac{1}{x^2} - \frac{2}{x^3}\right) dx = ?$

(a) $\frac{-e^x}{x^2} + C$ (b) $\frac{e^x}{x^2} + C$

(c) $e^x \left(\frac{-1}{x} + \frac{1}{x^2}\right) + C$ (d) none of these

45. $\int e^x \left\{\sin^{-1} x + \frac{1}{\sqrt{1-x^2}}\right\} dx = ?$

(a) $e^x \cdot \frac{1}{\sqrt{1-x^2}} + C$ (b) $e^x \sin^{-1} x + C$

(c) $\frac{-e^x}{\sin^{-1} x} + C$ (d) none of these

46. $\int e^x (\tan x + \log \sec x)\, dx = ?$

(a) $e^x \log \sec x + C$ (b) $e^x \tan x + C$

(c) $e^x (\log \cos x) + C$ (d) none of these

47. $\int e^x (\cot x + \log \sin x)\, dx = ?$

(a) $e^x \cot x + C$ (b) $e^x \log \sin x + C$

(c) $e^x \sin x + C$ (d) none of these

48. $\int e^x [\sec x + \log (\sec x + \tan x)] dx = ?$

(a) $e^x \log (\sec x + \tan x) + C$ (b) $e^x \sec x + C$

(c) $e^x \log \tan x + C$ (d) none of these

49. $\int e^x \left\{\tan^{-1} x + \frac{1}{(1+x^2)}\right\} dx = ?$

(a) $e^x \cdot \frac{1}{(1+x^2)} + C$ (b) $e^x \tan^{-1} x + C$

(c) $-e^x \cot^{-1} x + C$ (d) none of these

50. $\int e^x (\tan x - \log \cos x)\, dx = ?$

(a) $e^x \tan x + C$ (b) $e^x \log \cos x + C$

(c) $e^x \log \sec x + C$ (d) none of these

51. $\int e^x(\cot x - \text{cosec}^2 x)\,dx = ?$

(a) $-e^x \text{cosec}^2 x + C$ (b) $e^x \cot x + C$ (c) $-e^x \cot x + C$ (d) none of these

52. $\int e^x(\sin x + \cos x)\,dx = ?$

(a) $e^x \sin x + C$ (b) $e^x \cos x + C$ (c) $e^x \tan x + C$ (d) none of these

53. $\int e^x \sec x(1 + \tan x)\,dx = ?$

(a) $e^x(1 + \tan x) + C$ (b) $e^x \sec x + C$

(c) $e^x \tan x + C$ (d) none of these

54. $\int e^x \left(\frac{1 + x\log x}{x}\right) dx = ?$

(a) $e^x \cdot \frac{1}{x} + C$ (b) $e^x \log x + C$ (c) $x e^x \log x + C$ (d) none of these

55. $\int e^x \cdot \frac{x}{(1+x)^2}\,dx = ?$

(a) $e^x \cdot \frac{1}{(1+x)} + C$ (b) $e^x \cdot \frac{1}{x} + C$ (c) $e^x \cdot \frac{x}{(1+x)} + C$ (d) none of these

56. $\int e^x \left(\frac{1 + \sin x}{1 + \cos x}\right) dx = ?$

(a) $e^x \sin \frac{x}{2} + C$ (b) $e^x \cos \frac{x}{2} + C$ (c) $e^x \tan \frac{x}{2} + C$ (d) none of these

ANSWERS (OBJECTIVE QUESTIONS II)

1. (c) **2.** (b) **3.** (a) **4.** (b) **5.** (c) **6.** (c) **7.** (b) **8** (d) **9.** (b) **10.** (a)
11. (d) **12.** (d) **13.** (c) **14.** (c) **15.** (c) **16.** (b) **17.** (b) **18.** (b) **19.** (a) **20.** (b)
21. (b) **22.** (c) **23.** (d) **24.** (c) **25.** (b) **26.** (c) **27.** (b) **28.** (a) **29.** (b) **30.** (b)
31. (a) **32.** (b) **33.** (c) **34.** (b) **35.** (a) **36.** (b) **37.** (a) **38.** (b) **39.** (c) **40.** (a)
41. (b) **42.** (d) **43.** (c) **44.** (b) **45.** (b) **46.** (a) **47.** (b) **48.** (a) **49.** (b) **50.** (c)
51. (b) **52.** (a) **53.** (b) **54.** (b) **55.** (a) **56.** (c)

HINTS TO THE GIVEN OBJECTIVE QUESTIONS II

1. $\int \underset{\text{I}}{x}\ \underset{\text{II}}{e^x}\,dx$ 2. $\int \underset{\text{I}}{x}\ \underset{\text{II}}{e^{2x}}\,dx$ 3. $\int \underset{\text{I}}{x}\ \underset{\text{II}}{\cos 2x}\,dx$ 4. $\int \underset{\text{I}}{x}\ \underset{\text{II}}{\sec^2 x}\,dx$

5. $\int \underset{\text{I}}{x}\ \underset{\text{II}}{\sin 2x}\,dx$ 6. $\int \underset{\text{II}}{x}\ \underset{\text{I}}{\log x}\,dx$ 7. $\int \underset{\text{I}}{x}\ \underset{\text{II}}{\text{cosec}^2 x}\,dx$ 8. $I = \frac{1}{2}\int \underset{\text{I}}{x}\ \underset{\text{II}}{\sin 2x}\,dx$

9. $I = \int x \cdot \frac{1}{2}(1 + \cos 2x)\,dx = \frac{1}{2}\int x\,dx + \frac{1}{2}\int \underset{\text{I}}{x}\ \underset{\text{II}}{\cos 2x}\,dx$

10. $\int \frac{\log x}{x^2}\,dx = \int \underset{\text{I}}{(\log x)} \cdot \underset{\text{II}}{\frac{1}{x^2}}\,dx$ 11. $I = \int \underset{\text{I}}{(\log x)} \cdot \underset{\text{II}}{1}\,dx$

12. $I = \int \frac{\log x}{\log 10}\,dx = \frac{1}{\log_e 10} \cdot \int \underset{\text{I}}{(\log x)} \cdot \underset{\text{II}}{1}\,dx = (\log_{10} e) \cdot \int \left\{ \underset{\text{I}}{(\log x)} \cdot \underset{\text{II}}{1} \right\} dx$

13. $I = \left\{ \underset{\text{I}}{(\log x)^2} \cdot \underset{\text{II}}{1} \right\} dx = (\log x)^2 \cdot x - \int \frac{2\log x}{x} \cdot x\,dx$

$= x(\log x)^2 - 2\int \log x\,dx = x(\log x)^2 - 2[x(\log x - 1)] + C.$

14. Putting $\sqrt{x} = t$ and $\frac{1}{2\sqrt{x}}\,dx = dt$, we get $I = 2\int \underset{\text{I}}{t}\ \underset{\text{II}}{e^t}\,dt$

15. Putting $\sqrt{x} = t$ and $\frac{1}{\sqrt{x}}\,dx = 2dt$, we get $dx = 2t\,dt$

$\therefore \quad I = 2\int \underset{\text{I}}{t}\ \underset{\text{II}}{\cos t}\,dt.$

16. Putting $\log x = t$, $x = e^t$ and $dx = e^t dt$, we get

$$I = \int \underset{\text{I}}{e^t} \underset{\text{II}}{\cos t}\,dt = e^t \cos t + e^t \sin t - \int e^t \cos t\,dt$$

$\therefore \quad 2I = (e^t \cos t + e^t \sin t) \Rightarrow I = \frac{1}{2}(e^t \cos t + e^t \sin t) + C.$

17. $I = \int \underset{\text{II}}{\sec^2 x} \cdot \underset{\text{I}}{\sec x}\,dx = \sec x \tan x - \int \sec x \tan^2 x\,dx$

$= \sec x \tan x - \int \sec x(\sec^2 x - 1)\,dx = \sec x \tan x - I + \int \sec x\,dx$

$\therefore \quad 2I = \sec x \tan x + \log|\sec x + \tan x| + C.$

18. $I = \int \left\{ \underset{\text{I}}{\frac{1}{(\log x)}} \cdot \underset{\text{II}}{1} \right\} dx - \int \frac{1}{(\log x)^2}\,dx$

$= \frac{1}{(\log x)} \cdot x - \int (-1) \cdot \frac{1}{(\log x)^2} \cdot \frac{1}{x} \cdot x\,dx - \int \frac{1}{(\log x)^2}\,dx + C$

$= \frac{x}{(\log x)} + C.$

19. $I = \int (2x)x^2 e^{x^2}\,dx = \int \underset{\text{I}}{t}\ \underset{\text{II}}{e^t}\,dt$, where $x^2 = t$.

20. $I = \int \underset{\text{I}}{x} \cdot \underset{\text{II}}{2^x}\,dx = x \cdot \frac{2^x}{(\log 2)} - \int \frac{2^x}{(\log 2)}\,dx$

$= \frac{x \cdot 2^x}{(\log 2)} - \frac{2^x}{(\log 2)^2} + C$

21. $I = x(\operatorname{cosec}^2 x - 1)\,dx = \int \underset{\text{I}}{x}\ \underset{\text{II}}{\operatorname{cosec}^2 x}\,dx - \int x\,dx$

$= x(-\cot x) - \int (-\cot x)\,dx - \frac{x^2}{2} + C = -x\cos x + \log|\sin x| - \frac{x^2}{2} + C.$

22. Put $\sqrt{x} = t$ and $\frac{1}{2\sqrt{x}}\,dx = dt$ or $dx = 2t\,dt$

$\therefore \quad I = 2\int \underset{\text{I}}{t}\ \underset{\text{II}}{\sin t}\,dt.$

23. $I = 2\int e^{\sin x} \sin x \cos x\, dx = 2\int \underset{\text{I}}{t}\, \underset{\text{II}}{e^t}\, dt$, where $\sin x = t$.

24. Put $x = \sin t$ so that $dx = \cos t\ dt$ and $t = \sin^{-1} x$.

$$\therefore \qquad I = \int \frac{t \cos t}{(1 - \sin^2 t)^{3/2}}\, dt = \int \frac{t \cos t}{\cos^3 t}\, dt = \int \underset{\text{I}}{t}\, \underset{\text{II}}{\sec^2 t}\, dt.$$

25. Put $x = \tan t$, so that $dx = \sec^2 t\ dt$ and $t = \tan^{-1} x$.

$$\therefore \qquad I = \int \frac{t(\tan t)}{(1 + \tan^2 t)^{3/2}} \sec^2 t\, dt = \int \underset{\text{I}}{t}\, \underset{\text{II}}{\sin t}\, dt.$$

26. $I = \int \underset{\text{II}}{x} (\underset{\text{I}}{\tan^{-1} x})\, dx.$

27. Put $\sqrt{x} = t$ and $\frac{1}{2\sqrt{x}} dx = dt$ or $dx = 2t\ dt$.

$$\therefore \qquad I = 2\int \underset{\text{II}}{t}\, (\underset{\text{I}}{\tan^{-1} t})\, dt.$$

28. Put $\cos^{-1} x = t$, so that $x = \cos t$ and $dx = -\sin t\ dt$.

$$\therefore \qquad I = -\int \underset{\text{I}}{t}\, \underset{\text{II}}{\sin t}\, dt.$$

29. Put $\tan^{-1} x = t$, so that $x = \tan t$ and $dx = \sec^2 t\ dt$.

$$\therefore \qquad I = \int \underset{\text{I}}{t}\, \underset{\text{II}}{\sec^2 t}\, dt.$$

30. Put $\sec^{-1} x = t$, so that $x = \sec t$ and $dx = \sec t \tan t\ dt$.

$$\therefore \qquad I = \int \underset{\text{I}}{t} (\underset{\text{II}}{\sec t \tan t})\, dt.$$

31. Put $x = \sin t$ and $dx = \cos t\ dt$.

$$\therefore \qquad I = \int \sin^{-1}(\sin 3t) \cos t\ dt = 3\int \underset{\text{I}}{t}\, \underset{\text{II}}{\cos t}\, dt.$$

32. Put $x = \tan t$ and $dx = \sec^2 t\ dt$.

$$\therefore \qquad I = \int \sin^{-1}\left(\frac{2\tan t}{1 + \tan^2 t}\right) \sec^2 t\, dt = \int \sin^{-1}(\sin 2t) \sec^2 t\, dt$$

$$= 2\int \underset{\text{I}}{t}\, \underset{\text{II}}{\sec^2 t}\, dt.$$

33. Put $x = \cos t$ and $dx = -\sin t\ dt$. Then,

$$I = \int \tan^{-1} \sqrt{\frac{1 - \cos t}{1 + \cos t}} \cdot (-\sin t)\, dt = \int \tan^{-1} \sqrt{\frac{2\sin^2(t/2)}{2\cos^2(t/2)}} (-\sin t)\, dt$$

$$= -\int \tan^{-1}\left(\tan \frac{t}{2}\right)(-\sin t)\, dt = \frac{1}{2}\int \underset{\text{I}}{t}\, (\underset{\text{II}}{\sin t})\, dt.$$

34. Put $x = \tan t$ and $dx = \sec^2 t\ dt$.

$$I = \int \tan^{-1}(\tan 3t) \sec^2 t\, dt = 3\int \underset{\text{I}}{t}\, \underset{\text{II}}{\sec^2 t}\, dt.$$

35. $I = \int \underset{\text{I}}{x^2} \underset{\text{II}}{\cos x}\, dx = x^2 \sin x - \int \underset{\text{I}}{2x} \underset{\text{II}}{\sin x}\, dx$

$= (x^2 \sin x) - 2[x(-\cos x) - \int (-\cos x)\, dx] + C$

$= (x^2 \sin x) + 2x \cos x - 2 \sin x + C.$

36. Put $\cos x = t$ and $-\sin x\, dx = dt$.

$\therefore \quad I = -\int \log t\, dt = -\int \left(\underset{\text{I}}{\log t} \cdot \underset{\text{II}}{1} \right) dt = -\left[(\log t)t - \int \frac{1}{t} \cdot t\, dt \right]$

$= -t(\log t) + t + C = -\cos t[\log (\cos t)] + \cos x + C.$

37. $I = \frac{1}{2} \int \underset{\text{I}}{x} \underset{\text{II}}{\sin 2x}\, dx.$

38. Put $x^2 = t$, so that $x\, dx = \frac{1}{2} dt$.

$\therefore \quad I = \int \frac{1}{2} t \cos t\, dt = \frac{1}{2} \int \underset{\text{I}}{t} \underset{\text{II}}{\cos t}.$

39. Put $x = \tan t$ and $dx = \sec^2 t\, dt$.

Then, $I = \int \cos^{-1} \left(\frac{1 - \tan^2 t}{1 + \tan^2 t} \right) \sec^2 t\, dt = \int \cos^{-1}(\cos 2t) \sec^2 t\, dt$

$= 2 \int \underset{\text{I}}{t} \underset{\text{II}}{\sec^2 t}\, dt.$

40. $I = \int \underset{\text{II}}{x} \underset{\text{I}}{\tan^{-1} x}\, dx = (\tan^{-1} x) \cdot \frac{x^2}{2} - \int \frac{1}{(1 + x^2)} \cdot \frac{x^2}{2} dx$

$= \frac{1}{2} x^2 (\tan^{-1} x) - \frac{1}{2} \int \left(1 - \frac{1}{1 + x^2} \right) dx = \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} x + \frac{1}{2} \tan^{-1} x + C$

$= \frac{1}{2} (x^2 + 1) \tan^{-1} x - \frac{1}{2} x + C.$

41. $I = \int \left\{ \underset{\text{I}}{\sin (\log x)} \cdot \underset{\text{II}}{1} \right\} dx = (\sin \log x)x - \int \frac{\cos (\log x)}{x} \cdot x\, dx$

$= x \sin (\log x) - \int \left\{ \underset{\text{I}}{\cos (\log x)} \cdot \underset{\text{II}}{1} \right\} dx$

$= x \sin (\log x) - \cos (\log x) \cdot x + \int \frac{-\sin (\log x)}{x} \cdot x\, dx$

$= x \sin (\log x) - x \cos (\log x) - I$

$\therefore \quad 2I = x \sin (\log x) - x \cos (\log x) + C.$

42. Putting $x = \sin t$ and $dx = \cos t\, dt$, we get

$I = \int \{\sin^{-1}(\sin t)\}^2 \cos t\, dt = \int \underset{\text{I}}{t^2} \underset{\text{II}}{\cos t}\, dt$

$= t^2 \sin t - 2 \int \underset{\text{I}}{t} \underset{\text{II}}{\sin t}\, dt = t^2 \sin t - 2[t(-\cos t) - \int (-\cos t)\, dt]$

$= t^2 \sin t + 2t \cos t - 2 \sin t + C = x(\sin^{-1} x)^2 + 2(\sin^{-1} x)\sqrt{1 - x^2} - 2x + C.$

43. $I = \int e^x \{f(x) + f'(x)\}\, dx$, where $f(x) = \frac{1}{x}$

$= e^x f(x) + C = e^x \cdot \frac{1}{x} + C.$

44. $I = \int e^x \{f(x) + f'(x)\}\,dx$, where $f(x) = \dfrac{1}{x^2}$

$= e^x f(x) + C = e^x \cdot \dfrac{1}{x^2} + C.$

45. $I = \int e^x \{f(x) + f'(x)\}\,dx$, where $f(x) = \sin^{-1} x.$

46. $I = \int e^x \{f(x) + f'(x)\}\,dx$, where $f(x) = \log \sec x$

$= e^x f(x) + C = e^x \log \sec x + C.$

47. $I = \int e^x \{f(x) + f'(x)\}\,dx$, where $f(x) = \log \sin x$

$= e^x f(x) + C = e^x \log \sin x + C.$

48. $I = \int e^x \{f(x) + f'(x)\}\,dx$, where $f(x) = \log(\sec x + \tan x)$

$= e^x f(x) + C = e^x \log(\sec x + \tan x) + C.$

49. $I = \int e^x \{f(x) + f'(x)\}\,dx$, where $f(x) = \tan^{-1} x$

$= e^x f(x) + C = e^x \tan^{-1} x + C.$

50. $I = \int e^x \{f(x) + f'(x)\}\,dx$, where $f(x) = \log \sec x$

$= e^x f(x) + C = e^x \log \sec x + C.$

51. $I = \int e^x \{f(x) + f'(x)\}\,dx$, where $f(x) = \cot x$

$= e^x f(x) + C = e^x \cot x + C.$

52. $I = \int e^x \{f(x) + f'(x)\}\,dx$, where $f(x) = \sin x$

$= e^x f(x) + C = e^x \sin x + C.$

53. $I = \int e^x \{f(x) + f'(x)\}\,dx$, where $f(x) = \sec x$

$= e^x \sec x + C.$

54. $I = \int e^x \left(\dfrac{1}{x} + \log x\right) dx$

$= \int e^x \{f(x) + f'(x)\}\,dx$, where $f(x) = \log x$

$= e^x f(x) + C = e^x \log x + C.$

55. $I = \int e^x \left\{\dfrac{(1+x)-1}{(1+x)^2}\right\} dx = \int e^x \left\{\dfrac{1}{(1+x)} - \dfrac{1}{(1+x)^2}\right\} dx$

$= \int e^x \{f(x) + f'(x)\}\,dx$, where $f(x) = \dfrac{1}{(1+x)}$

$= e^x f(x) + C = e^x \cdot \dfrac{1}{(1+x)} + C.$

56. $I = \int e^x \left(\dfrac{1 + 2\sin(x/2)\cos(x/2)}{2\cos^2(x/2)}\right) dx = \int e^x \left(\tan\dfrac{x}{2} + \dfrac{1}{2}\sec^2\dfrac{x}{2}\right) dx$

$= \int e^x \{f(x) + f'(x)\}\,dx$, where $f(x) = \tan\dfrac{x}{2}$

$= e^x f(x) + C = e^x \tan\dfrac{x}{2} + C.$